1. Did I understand it?

These questions are not to be handed in; they are meant as recommended (relatively easy) extra exercise.

(1) Show that if $f$ is continuous at $a$ then $f(x_n) \to f(a)$ for any sequence $x_n \to a$.

Proof. Indeed, let $f$ be a continuous function and suppose $x_n \to a$. Let $\epsilon > 0$. Then there exists a $\delta$ such that for all $x$ with $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Given $\delta$ there exists an $N_0$ such that for all $n > N_0$ we have $|x_n - a| < \delta$, and thus also $|f(x_n) - f(a)| < \epsilon$.

(2) Show that if $\{x_n\}$ and $\{y_n\}$ are sequences such that the compound sequence $\{z_n\} = x_1, y_1, x_2, y_2, x_3, \ldots$, that is $z_n = x_{(n+1)/2}$ for odd $n$ and $z_n = x_n/2$ for even $n$, converges $(z_n \to l)$ if and only if $x_n \to l$ and $y_n \to l$.

Proof. Suppose $z_n$ converges to $l$. Let $\epsilon > 0$. Then there exists an $N_0$ such that for all $n > N_0$ we have $|z_n - l| < \epsilon$. Thus for all $n > N_0/2$ we have both $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$; that is, $x_n \to l$ and $y_n \to l$.

Now suppose $x_n \to l$ and $y_n \to l$. Let $\epsilon > 0$. Then there exist $N_0$ and $N_1$ such that for all $n > N_0$ we have $|x_n - l| < \epsilon$ and for all $n > N_1$ we have $|y_n - l| < \epsilon$. In particular we have for all $n > 2 \max(N_0, N_1)$ that $|z_n - l| < \epsilon$. So we conclude that $z_n \to l$.

(3) Determine the lim sup and lim inf of the sequence $a_n = \frac{1}{n} + 2^{\cos(n)}$.

Proof. $\limsup a_n = 2$ and $\liminf a_n = 1/2$.

(4) Show that $\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$, unless the right hand side says $\infty$ or $(-\infty)$ (for other options with $\pm \infty$ on the right hand side we use the “obvious” addition rules). Give an example where the inequality is strict.

Proof. If either $a_n$ or $b_n$ is unbounded then this equation is trivial. Otherwis, note that $\sup\{a_m : m > n\} + \sup\{b_m : m > n\} \geq a_s + b_s$ for any $s > n$ (by definition of supremum). In particular it is an upperbound to $\{a_m + b_m : m > n\}$, and thus we get

$$\sup\{a_m : m > n\} + \sup\{b_m : m > n\} \geq \sup\{a_m + b_m : m > n\}.$$ 

Now take the limit in this inequality.

An example where the inequality is strict is taking $a_n = (-1)^n$ and $b_n = -a_n$.

(5) Show that $\limsup a_n = -\liminf -a_n$.

Proof. Indeed, using $\sup(A) = -\inf\{-a : a \in A\}$, which follows as if $x$ is an upperbound for $A$, then $-x$ is an lowerbound for $\{-a : a \in A\}$, and vice versa, we have

$$\limsup a_n = \lim_{n \to \infty} \sup\{a_m : m > n\} = \lim_{n \to \infty} -\inf\{-a_m : m > n\} = -\liminf -a_n$$

(6) Give an example of a non-convergent Cauchy sequence in $\mathbb{Q}$.

Proof. 1, 1.4, 1.41, 1.414, 1.4142, etc.