Let $R$ be a ring with identity $1 \neq 0$, and $M$ a left $R$-module.

1) a) The endomorphism ring of $R$-module homomorphisms $f : M \to M$ is defined as $\text{End}_R(M)$. The opposite ring $R^{\text{op}}$ of $(R, \cdot, +)$ is defined as $(R, \cdot^{\text{op}}, +)$, where $a \cdot^{\text{op}} b = b \cdot a$. Show that there is an isomorphism of rings $\text{End}_R(R^{\oplus n}) \cong \text{Mat}_n(R^{\text{op}})$.

b) Show that the center of the ring $\text{Mat}_n(R)$ is isomorphic to the center of the ring $R$.

2) Let $R$ be a left Artinian ring, and $J(R)$ be its Jacobson radical, the intersection of all maximal left ideals of $R$.

a) Show that $J(R)$ is nilpotent (i.e. $J(R)^n = 0$ for some integer $n \geq 1$).

(Hint: $J(R)^r$ form a descending chain. Consider the ideal $J(R)^\infty = \bigcap_{r=1}^{\infty} J(R)^r$. Show that it is zero by applying Nakayama’s lemma on a minimal non-zero finitely generated submodule $L$ of $J(R)^\infty$ and show a contradiction.)

b) Prove the generalization of Nakayama’s lemma for non-commutative rings: If $M$ is a left $R$-module and $J(R)M = M$, then $M = 0$.

c) For a left Artinian ring $R$, show that the following are equivalent:

i) $M$ is left Artinian,

ii) $M$ is left Noetherian,

iii) $M$ is finitely generated.

d) Show that $R$ is a left Artinian ring $\Rightarrow$ $R$ is a left Noetherian ring, and that any finitely generated $R$-module admits a Jordan-Hölder filtration (i.e. of finite length).