#7.1.10(b) Compute the path integral of \( f(x, y, z) = \cos(z) \) over the path \( c(t) = (\cos(t), \sin(t), t), 0 \leq t \leq \frac{\pi}{2} \).

**Solution.**

We have \( c'(t) = (-\sin(t), \cos(t), 1) \), hence

\[
||c'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} = \sqrt{2}.
\]

Therefore

\[
\int_{c} f(x, y, z) ds = \int_{0}^{\pi/2} \cos(t) \sqrt{2} dt = \sqrt{2}.
\]

#7.1.24. Compute the path integral of \( f(x, y) = y^2 \) over the graph of \( y = e^x, x \in [0, 1] \).

**Solution.**

We interpret this graph as the path \( \mathbf{c}(t) = (t, e^t) \), where \( t \) ranges from 0 to 1. Recall the formula for computing path integrals:

\[
\int_{c} f(x, y) ds = \int_{0}^{1} f(\mathbf{c}(t)) \cdot \sqrt{(c'_1)^2 + (c'_2)^2} \, dt
\]

\[
= \int_{0}^{1} e^{2t} \cdot \sqrt{1 + e^{2t}} \, dt.
\]

So, we can use the substitution \( u = 1 + e^{2t}, du = 2e^{2t} \) to get

\[
\int_{c} f(x, y) ds = \int_{2}^{1+e^2} \frac{\sqrt{u}}{2} \, du
\]

\[
= \frac{u^{3/2}}{3} \bigg|_{2}^{1+e^2}
\]

\[
= \frac{(1 + e^2)^{3/2} - \sqrt{8}}{3}.
\]

#7.2.4(c) Let \( \mathbf{c}_1 \) be the straight line path from \((1, 0, 0)\) to \((0, 1, 0)\), which we can parametrize by \((1 - t, t, 0), 0 \leq t \leq 1\). Let \( \mathbf{c}_2 \) be the straight line path from \((0, 1, 0)\) to \((0, 0, 1)\), which we can parametrize by \((0, 1 - t, t), 0 \leq t \leq 1\).

We have

\[
\int_{c_1} yz \, dx + xz \, dy + xy \, dz = 0
\]

since \( z = 0 \) and hence \( \frac{dz}{dt} = 0 \) on \( c_1 \).

Similarly

\[
\int_{c_2} yz \, dx + xz \, dy + xy \, dz = 0
\]
since \( x = 0 \) and hence \( \frac{dx}{dt} = 0 \) on \( c_2 \). Therefore
\[
\int_{c} yz \, dx + xz \, dy + xy \, dz = 0.
\]

### 7.2.11

The image of the path \( c(t) = (\cos^3(t), \sin^3(t)), t \in [0, 2\pi] \) in the plane is illustrated below. Evaluate the integral of the vector field \( \mathbf{F}(x, y) = (x, y) \) around this curve.

**Solution.**

We have
\[
\int_{c} \mathbf{F} \cdot ds = \int_{0}^{2\pi} \mathbf{F}(c(t)) \cdot \mathbf{c}'(t) \, dt
\]
\[
= \int_{0}^{2\pi} (\cos^3(t), \sin^3(t)) \cdot (-3 \cos^2(t) \sin(t), 3 \sin^2(t) \cos(t)) \, dt
\]
\[
= \int_{0}^{2\pi} (-3 \cos^5(t) \sin(t) + 3 \sin^5(t) \cos(t)) \, dt
\]
\[
= \left( \frac{1}{2} \cos^6(t) \right)^{2\pi}_{0} + \left( \frac{1}{2} \sin^6(t) \right)^{2\pi}_{0}
\]
\[
= 0
\]

### 7.3.9

The parametrization given is exactly the standard spherical coordinates, so the surface is a sphere. Write \( T(u, v) = (\cos v \sin u, \sin v \sin u, \cos u) \). Then we have \( T_u(u, v) = (\cos v \cos u, \sin v \cos u, -\sin u) \) and \( T_v(u, v) = (-\sin v \sin u, \cos v \sin u, 0) \). The unit normal is
\[
\frac{T_u \times T_v}{||T_u \times T_v||} = (\cos v \sin u, \sin v \sin u, \cos u).
\]

This is geometrically natural, since the unit normal to a point on a sphere is parallel to the vector from the origin to that point.

### 7.3.14

Find the equation for the plane tangent to the surface \( x = u^2, y = v^2, z = u^2 + v^2 \) at the point \((u, v) = (1, 1)\).

**Solution.** Our surface is parametrized by the map \( T(u, v) = (u^2, v^2, u^2 + v^2) \). Therefore, two vectors tangent to our surface at any point are given by
\[
\frac{\partial T}{\partial u} = (2u, 0, 2u), \quad \frac{\partial T}{\partial v} = (0, 2v, 2v).
\]

These vectors are linearly independent: therefore, they span our tangent plane. Therefore, if we take their cross-product, we will get a vector orthogonal to our tangent plane:
\[
\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} = (-4uv, -4uv, 4uv).
\]
At \((u, v) = (1, 1)\), this is the vector \((-4, -4, 4)\). So, we want a plane made out of vectors orthogonal to \((-4, -4, 4)\) through the point \(T(1, 1) = (1, 1, 2)\): i.e. we want the collection of all points \((x, y, z)\) such that
\[
(x - 1, y - 1, z - 2) \cdot (-4, -4, 4) = 0.
\]

**#7.4.6** The is the area of the graph of the function \(z = xy\) over a disc \(D\) of radius 2 centered at the origin. We have \(\frac{dz}{dx} = x\) and \(\frac{dz}{dy} = y\). The area element is \(r\), so using the formula for surface area, we have
\[
\text{Area} = \int \int_D \sqrt{1 + x^2 + y^2} \, dA
= \int_0^{2\pi} \int_0^2 r \cdot \sqrt{1 + r^2} \, dr \, d\theta
= \frac{2\pi}{3} (\sqrt{125} - 1)
\]

**#7.4.15.** Find the area of the surface obtained by rotating the curve \(y = x^2, 0 \leq x \leq 1\) about the \(y\)-axis.

**Solution.** Formula (6) on page 388 of the text gives the area \(A\) of the surface obtained by rotating the graph of a function \(f\) of \(x, a \leq x \leq b\) around the \(y\)-axis as
\[
A = 2\pi \int_a^b (|x|\sqrt{1 + (f'(x))^2}) \, dx
\]. So in our case,
\[
A = 2\pi \int_0^1 x\sqrt{1 + 4x^2} \, dx
= 2\pi \left(\frac{1}{12}(4x^2 + 1)^{\frac{3}{2}}\right)^1_0
= \frac{(\sqrt{125} - 1)\pi}{6}.
\]