1. Problem 2.1.5

(a)

(b)
2. Problem 2.1.10

(a) Since \( f(x, y) \geq 1 \) for all \( x, y \in \mathbb{R}^2 \), we will have non-empty level curves only when \( c \geq 1 \). These are of the form \( x^2 + y^2 + 1 = c \), or \( x^2 + y^2 = c - 1 \). Thus level curves are circles centered around the origin, with radius \( \sqrt{c-1} \). Increasing \( c \) will increase the radius.

(b) Similar to part (a). Since \( f(x, y) \leq 1 \) for all \( x, y \in \mathbb{R}^2 \), only values of \( c \) where \( c \leq 1 \) produce non-empty level curves. These have form \( x^2 + y^2 = 1 - c \), circles around the origin with radius \( \sqrt{1-c} \). Decreasing \( c \) will increase the radius.

(c) For a fixed \( c \), the level curve \( f(x, y) = x^3 - x = c \) will consist of a union of vertical lines \( x = x_i \), \( 1 \leq i \leq k \), where \( k \) is the number of roots of the polynomial \( x^3 - x - c \), where \( x_i \) are the roots. We can obtain a visual representation of the situation by flipping the graph of \( g(x) = x^3 - x \):

We can work out the local maximum and minimum of \( x^3 - x \); they have values \( 2\sqrt{3}/9 \) and \( -2\sqrt{3}/9 \), respectively. Thus when \( c \in (-2\sqrt{3}/9, 2\sqrt{3}/9) \), we will have three roots for \( x^3 - x - c = 0 \), and so our level curve will consist of three vertical lines. For \( c \) on the boundaries of this interval, we will have two roots, and outside this interval only one. Outside this interval, increasing \( c \) shifts our vertical line to the right.
2  Problem 2.1.10

(a) $f(x, y) = x^2 + y^2 + 1$
(b) $f(x, y) = 1 - x^2 - y^2$
(c) $f(x, y) = x^3 - x$
3 Problem 2.1.27

$4x^2 + y^2 = 16$
4 Problem 2.2.4

(a) Polynomial and exponential functions are continuous so limit exists at all points for them. Their composition is also continuous. Therefore \(\lim_{(x,y)\to(0,1)} e^x = e^0 = 1\) (composition of exponential and polynomial) and \(\lim_{(x,y)\to(0,1)} y = 1\). Using Theorem 3(iii) from the book we know that

\[ \lim_{(x,y)\to(0,1)} e^x y = \lim_{(x,y)\to(0,1)} e^x \lim_{(x,y)\to(0,1)} y = 1. \]

(b) The conditions for L’Hospital’s rule are satisfied and thus

\[ \lim_{x\to0} \frac{\sin^2 x}{x} = \lim_{x\to0} 2 \sin x \cos x = 0. \]

(c)\(\lim_{x\to0} \frac{(\sin x)^2}{x} = \lim_{x\to0} \frac{\sin x}{x} \lim_{x\to0} \frac{\sin x}{x} = 1.\)

Using the fact that \(\lim_{x\to0} \frac{\sin x}{x} = 1\) and Theorem 3(iii).

5 Problem 2.2.12

(a) The conditions for L’Hospital’s rule are satisfied and thus the limit exists. Using L’Hospital’s rule a number of times, we get

\[ \lim_{x\to0} \frac{\sin 2x - 2x^3}{x^3 + y} = \lim_{x\to0} \frac{2 \cos 2x - 2}{3x^2} = \lim_{x\to0} \frac{-4 \sin 2x}{6x} = \lim_{x\to0} \frac{-8 \cos 2x}{6} = -\frac{4}{3}. \]

(b) We will take limits along two directions first with \(x = 0\) and then with \(y = 0\). Along \(x = 0\) we get \(\lim_{(x,y)\to(0,0)} \frac{\sin 2x - 2x^3 + y}{x^3 + y} = \lim_{y\to0} \frac{y}{y} = 1\). Along \(y = 0\) we get \(\lim_{(x,y)\to(0,0)} \frac{\sin 2x - 2x^3 + y}{x^3 + y} = \lim_{x\to0} \frac{\sin 2x - 2x^3}{x^3} = -\frac{4}{3}\) by part (a). If the limit existed, along both these directions it should have been same. So the limit does not exist.

(c)By example 15(b), we know \(\lim_{(x,y)\to(0,0)} \frac{2x^2 y \cos z}{x^2 + y^2} = 0.\) And we know \(\lim_{z\to0} \cos z = 1.\) So \(\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 y \cos z}{x^2 + y^2} = \lim_{(x,y)\to(0,0,0)} \frac{2x^2 y}{x^2 + y^2} \lim_{(x,y,z)\to(0,0,0)} \cos z = 0\) by Theorem 3(iii).