Problem 2.
Let $A\mathbf{x} = \mathbf{b}$ be such a system. Then $A\mathbf{x} = \mathbf{0}$ must have solution set $s(1,2,1)^T$. Note that there is exactly one free variable; one possibility for $A$ is
\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -2
\end{pmatrix}.
\]
Now $(1,1,0)^T$ is a solution to $A\mathbf{x} = \mathbf{b}$, so for this choice of $A$ we can compute $\mathbf{b} = (1,1)^T$.

Problem 3.
The matrix $A_1$ is row equivalent to
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]
There are two pivots, so $A_1$ has rank 2 and nullity 1. A basis for the range would be the first two columns of $A_1$, $(1,0,1)^T$ and $(1,1,1)^T$. Solving $A_1\mathbf{x} = \mathbf{0}$ gives $\mathbf{x} = x_3(0,-1,1)^T$, so $\{(0,-1,1)^T\}$ is a basis for the kernel. $A_2$, in reduced row echelon form, is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1/4 & 1/2 \\
0 & 0 & 1 & 1/6 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
From this, we deduce that $A_2$ has rank 3, nullity 2, $\{(1,1,0,1)^T, (2,4,2,0)^T, (3,0,-3,0)^T\}$ as a basis for its range, and $\{(0,-1/4,-1/6,1,0)^T, (0,-1/2,0,0,1)^T\}$ as a basis for its kernel.

Problem 4.
We can row reduce the given matrix to echelon form:
\[
\begin{pmatrix}
1 & 2 & -1 & 2 & 3 \\
0 & -2 & 3 & 1 & -1 \\
0 & 0 & 0 & 2 & -5 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
The pivots are in columns 1, 2, and 4, so the corresponding columns in the original matrix form a basis for the column space: $\{(1,2,3,-1)^T, (2,2,6,-4)^T, (2,5,0,-7)^T\}$. The nonzero rows of the echelon form matrix form a basis for the row space: $\{(1,2,-1,2,3)^T, (0,-2,3,1,-1)^T, (0,0,0,2,-5)^T\}$.

Problem 5.
Let $\mathcal{A} = \{(1,0)^T, (0,1)^T\}$ be the standard basis and $\mathcal{B} = \{(1,1)^T, (1,2)^T\}$. Then
\[
[I]_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad [I]_{\mathcal{B}\mathcal{A}} = ([I]_{\mathcal{A}\mathcal{B}})^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.
\]
We can read off the matrix for $T$ in the standard basis, and then compute its representation in the basis $\mathcal{B}$ via the change-of-basis matrices:
\[
[T]_{\mathcal{A}\mathcal{A}} = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}, \quad [T]_{\mathcal{B}\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}[T]_{\mathcal{A}\mathcal{A}}[I]_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 9 & 13 \\ -5 & -8 \end{pmatrix}.
\]
Problem 6.
(a) Note that the condition we want to show is equivalent to $T$ being an isomorphism. It was shown in Problem 6 of HW2 that the three columns of this matrix are linearly independent. Thus the column space has dimension 3, and so $T$ has rank 3. The domain and codomain are both 3-dimensional, and thus $T$ is an isomorphism.

(b) We wish to show that $T$ is an isomorphism. Since the domain and codomain have the same dimension, it suffices to show that $T$ has trivial kernel. Consider some $f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$, $m \leq n$, $a_m \neq 0$. This has leading term $a_m x^m$. Now

$$(T f)(x) = x(x - 1)(m(m - 1)a_m x^{m-2} + \cdots) + (x - 1)(ma_m x^{m-1} + \cdots) + (a_m x^m + \cdots).$$

Observe that this has leading term $a_m (m(m - 1) + m + 1)x^m = a_m (m^2 + 1)x^m$, since $m^2 + 1 \neq 0$ always. In particular, $(T f)(x)$ and $f(x)$ will always have the same degree. Thus, if $(T f)(x) = 0$, $f(x)$ must be a constant polynomial. But if $f(x)$ is constant, then $f'(x) = f''(x) = 0$, and so $(T f)(x) = f(x) = 0$. Thus 0 is the only member of the kernel of $T$, and thus $T$ is an isomorphism.