**Solutions to Homework 1**

**Problem 2.**

1. Not a subspace—the subset contains $v = (1, 1, \ldots, 1)^T$ but not $2v = (2, 2, \ldots, 2)^T$.

2. This is a subspace of $\mathbb{F}^n$. Take any two members, $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$, and any $c \in \mathbb{F}$. Then $cx + y = (cx_1 + y_1, \ldots, cx_n + y_n)^T$, and
   $$(cx_1 + y_1) + \cdots + (cx_n + y_n) = c(x_1 + \cdots + x_n) + (y_1 + \cdots + y_n) = cnx_n + ny_n = n(cx_n + y_n).$$
   Thus $cx + y$ also lies in the subset, so it is closed under addition and scalar multiplication.

3. In the case $\mathbb{F} = \mathbb{R}$, observe that the subset contains only one element, the zero vector (a sum of nonnegative real numbers is 0 iff all the summands are 0), and so it is the trivial subspace. In the case $\mathbb{F} = \mathbb{C}$, this subset is not a subspace. For instance, it contains $v = (1, i, 0, \ldots, 0)^T$ and $w = (i, 1, 0, \ldots, 0)^T$, but not $v + w = (1 + i, 1 + i, 0, \ldots, 0)^T$, since $(1 + i)^2 + (1 + i)^2 = 4i \neq 0$.

4. Suppose we have $v$ and $w$ in the subset $\left\{ (x_1, \ldots, x_n)^T \in \mathbb{R}^n \mid \sum_{i=1}^n c_i x_i = 0 \right\}$, for some $c_i \in \mathbb{R}$. Say $v = (v_1, \ldots, v_n)^T$ and $w = (w_1, \ldots, w_n)^T$. Then for any $c \in \mathbb{R}$, $cv + w = (cv_1 + w_1, \ldots, cv_n + w_n)^T$, and
   $$\sum_{i=1}^n c_i (cv_i + w_i) = c \sum_{i=1}^n c_i v_i + \sum_{i=1}^n c_i w_i = 0,$$
   so $cv + w$ also lies in the subset. Observe now that the three constraints in the subset defined in the problem are all of the form $\sum_{i=1}^n c_i x_i = 0$ (here $n = 5$). Applying this argument, for any $v$ and $w$ in the given subset and for any $c \in \mathbb{R}$, $cv + w$ satisfies all three constraints and thus lies in the subset also. We have shown that the subset is a subspace.

**Problem 3.**

One possible basis is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$ 

This contains three elements.

**Problem 4.**

(a) \[
\begin{pmatrix} 13 \\ 31 \end{pmatrix}
\]

(c) \[
\begin{pmatrix} 5 \\ 8 \\ 11 \\ 4 \end{pmatrix}
\]

**Problem 5.**

(a) The matrix for $T$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{pmatrix}$$
(b) For convenience, we first compute the matrices for the maps $F : f(t) \mapsto f'(t)$ and $G : f(t) \mapsto f''(t)$:

$$[F] = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad [G] = \begin{pmatrix} 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$ 

Now $T = 2I + 3F - 4G$, so we have

$$[T] = \begin{pmatrix} 2 & 3 & -8 & \cdots & 0 \\ 0 & 2 & 6 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -4n(n-1) \\ 0 & 0 & 0 & \cdots & 3(n-2) \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$ 

**Problem 6.**

$$AB = \begin{pmatrix} 18 & 6 & -5 \\ 19 & -2 & 20 \end{pmatrix}, \quad A(3B + C) = \begin{pmatrix} 10 & 5 \\ 3 & 1 \end{pmatrix}, \quad B^T A = \begin{pmatrix} 10 & 5 \\ -4 & 2 \end{pmatrix}.$$ 

Here we see a demonstration of the associativity of matrix multiplication:

$$BD = \begin{pmatrix} 0 \\ -6 \end{pmatrix}, \quad A(BD) = (AB)D = \begin{pmatrix} -12 \\ -6 \end{pmatrix}.$$ 

**Problem 7.**

One example is the following:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Indeed, we have

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$