1. (5 pts) [Apostol 5.5.9] Let $V$ be a complex inner product space and let $T : V \rightarrow V$ be a linear transformation. Define $Q(x) = (T(x), x)$. Prove the following:

(a) If $T$ is Hermitian, then $Q(x)$ is real for all $x$.
(b) If $T$ is skew-Hermitian, then $Q(x)$ is purely imaginary for all $x$.
(c) $Q(tx) = t\overline{Q(x)}$ for all scalars $t$.
(d) $Q(x + y) = Q(x) + Q(y) + (T(x), y) + (T(y), x)$. Find a corresponding formula for $Q(x + ty)$.
(e) If $Q(x) = 0$ for all $x$, then $T(x) = 0$ for all $x$.
(f) If $Q(x)$ is real for all $x$, then $T$ is Hermitian.

Proof. For part (a), if $T$ is Hermitian then

$$Q(x) = (T(x), x) = (x, T(x)) = \overline{(T(x), x)} = \overline{Q(x)}.$$ 

This implies $Q(x) \in \mathbb{R}$.

For part (b), if $T$ is skew-Hermitian, then

$$Q(x) = (T(x), x) = -(x, T(x)) = -\overline{(T(x), x)} = -Q(x).$$ 

This implies $Q(x) \in i\mathbb{R}$.

For part (c), we have $Q(tx) = (T(tx), tx) = (tT(x), tx) = t(T(x), tx) = t\overline{T(x), x} = t\overline{Q(x)}$.

For part (d), we have

$$Q(x + y) = (T(x + y), x + y) = (T(x) + T(y), x + y) = (T(x), x) + (T(x), y) + (T(y), x) + (T(y), y) = Q(x) + Q(y) + (T(x), y) + (T(y), x)$$

and

$$Q(x + ty) = Q(x) + Q(ty) + (T(x), ty) + (T(ty), x) = Q(x) + t\overline{Q(y)} + \overline{T(tx), y} + t(T(y), x)$$

For part (e), we use the formula in part (d):

$$Q(x + ty) = Q(x) + t\overline{Q(y)} + \overline{T(tx), y} + t(T(y), x).$$

If $Q(x) = 0$ for all $x$, then

$$0 = \overline{T(tx), y} + t(T(y), x)$$
for all scalars $t$. Choosing $t = 1$ gives

$$0 = (T(x), y) + (T(y), x)$$

and choosing $t = i$ gives

$$0 = -(T(x), y) + (T(y), x).$$

Therefore, $(T(x), y) = 0$ for all $x, y \in V$, which implies $T(x) = 0$ for all $x$.

For part (f), we again use the formula from (d)

$$Q(x + ty) = Q(x) + tQ(y) + t(T(x), y) + t(T(y), x).$$

Taking conjugates gives

$$Q(x + ty) = Q(x) + tQ(y) + t(y, T(x)) + t(x, T(y)).$$

If $Q$ is real valued, then these are equal. This gives

$$t(T(x), y) + t(T(y), x) = t(y, T(x)) + t(x, T(y)).$$

Taking $t = 1$ and $t = i$ again gives a system of equations which can be solved, yielding

$$(T(x), y) = (x, T(y)).$$

\[\square\]

2. (3 pts) Let $L_{a,b}$ in $\mathbb{R}^2$ be the straight line defined by the equation $y = a + bx$ for some $a, b \in \mathbb{R}$. Given a collection of points $S = \{(x_i, y_i)\}_{i=1}^k$ in $\mathbb{R}^2$, a line is said to best fit $S$ if it minimizes the quadratic error $\sum_{i=1}^k |a + bx_i - y_i|^2$. (e.g. this would be the line passing through all points of $S$ if it existed.

Find the best straight line fit (least square solution) to the points $(-2,4), (-1,3), (0,1), (2,0)$.

Proof. Let

$$f(a,b) = \sum_{i=1}^n (a + bx_i - y_i)^2$$

Then

$$\frac{\partial f}{\partial a} = \sum_{i=1}^n 2(a + bx_i - y_i)$$

and

$$\frac{\partial f}{\partial b} = \sum_{i=1}^n 2(a + bx_i - y_i)x_i$$

Setting the partial derivatives to 0, one obtains the following system of linear equations

$$na + \left(\sum_{i=1}^n x_i\right)b = \sum_{i=1}^n y_i$$

$$\left(\sum_{i=1}^n x_i\right)a + \left(\sum_{i=1}^n x_i^2\right)b = \sum_{i=1}^n x_iy_i$$

In our case, the matrix for the system is

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \end{pmatrix}$$
Solving this, one obtains that $a = \frac{61}{35}$ and $b = -\frac{36}{35}$. One can check this indeed is a minimum.

Below is a sketch of a solution without use of multivariable calculus:

Suppose given $(x_1, y_1), ..., (x_n, y_n)$. In $\mathbb{R}$, let $\bar{1} = (1, ..., 1), \bar{x} = (x_1, ..., x_n)$, and $\bar{y} = (y_1, ..., y_n)$. One seeks to find $a$ and $b$ such that $||a\bar{1} + b\bar{x} - \bar{y}||$ is smallest possible. Let $S$ be the space spanned by $\bar{1}$ and $\bar{x}$. Then $a\bar{1} + b\bar{x} = \text{proj}_S(\bar{y})$. Hence one can calculate the projection and write it terms of the basis $\bar{1}$ and $\bar{x}$ to find $a$ and $b$.

\[ \Box \]

3. (5 pts) Let $v_1 = (1, 0, 1, 0)^T$, $v_2 = (2, 2, 0, 0)^T$, and $V = \langle v_1, v_2 \rangle$ in $\mathbb{R}^4$.

(a) Find $\{u_1, u_2\}$ an orthogonal basis of $V$.

(b) For $i = 1, 2$: express $v_i$ as a linear combination of the new basis.

(c) Compute the orthogonal complement of $V$.

(d) Complete $\{u_1, u_2\}$ to an orthogonal basis of $\mathbb{R}^4$.

(e) For $w = (1, -1, -1, 0)^T$: does $w$ belong to the space $V$? If possible, write $w$ as a linear combination of $u_1, u_2$.

(f) For $w = (1, 1, 1, 1)^T$: does $w$ belong to the space $V$? If not, find the distance from $w$ to $V$.

**Solution.** (a) Notice that $(v_1, v_2) = (v_1, v_1) = 2$, so $(v_1, v_2 - v_1) = 0$. In other words, $u_1 = v_1 = (1, 0, 1, 0)^T$ and $u_2 = v_2 - v_1 = (1, 2, -1, 0)^T$ forms an orthogonal basis of $V$.

(b) Since $u_1 = v_1$ and $u_2 = v_2 - v_1$, so $v_1 = u_1$ and $v_2 = u_1 + u_2$.

(c) Let $v$ be an element in the orthogonal complement of $V$, then

\[ (v, v_1) = (v, v_2) = 0. \]

Denote $v = (a, b, c, d)^T$, then $a + c = 0$ and $a + b = 0$. So we get

\[ v = (a, -a, -a, d)^T = a(1, -1, -1, 0)^T + d(0, 0, 0, 1)^T. \]

Therefore the orthogonal complement is generated by $(1, -1, -1, 0)^T$ and $(0, 0, 0, 1)^T$.

(d) Denote $u_3 = (1, -1, -1, 0)^T$ and $u_4 = (0, 0, 0, 1)^T$, then $u_3, u_4$ generates the orthogonal complement of $V$. Moreover, $(u_3, u_4) = 0$, so $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis of $\mathbb{R}^4$.

(e) Notice that $w = u_3$ is a non-zero vector in the orthogonal complement of $V$, so $w$ does not belong to $V$.

(f) Denote $V^\perp$ the orthogonal complement of $V$. Let $v$ be the projection of $w$ on $V^\perp$. Then $w \in V$ if and only if $v = 0$. Since $\{u_3, u_4\}$ is an orthogonal basis of $V^\perp$, so $v$ the projection of $w$ on $V^\perp$ satisfies

\[ v = \frac{(w, u_3)}{(u_3, u_3)}u_3 + \frac{(w, u_4)}{(u_4, u_4)}u_4. \]

We plug in $w, u_3, u_4$ in the above equation and obtain that

\[ v = \frac{1}{3} u_3 + u_4. \]
As \( v \neq 0 \), so \( w \) does not belong to \( V \). Moreover, the distance from \( w \) to \( V \) is exact \( ||v|| \). Since \( u_3 \perp u_4 \), so
\[
(v, v) = (-\frac{1}{3}u_3, -\frac{1}{3}u_3) + (u_4, u_4) = \frac{1}{9}||u_3||^2 + ||u_4||^2 = \frac{1}{3} + 1 = \frac{4}{3}.
\]
Then \( ||v|| = \frac{2\sqrt{2}}{3} \) is the distance from \( w \) to \( V \).

\[ \square \]

4. (7 pts) [Apostol, 5.11, Problems 7 and 13] **Problem 7**: Find (a) an orthogonal set of eigenvectors for \( A \), and (b) a unitary matrix \( C \) such that \( C^{-1}AC \) is a diagonal matrix.

\[
A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.
\]

**Problem 13**: If \( A \) is real skew-symmetric, prove that both \( I \pm A \) are nonsingular and \( (I - A)(I + A)^{-1} \) is orthogonal.

**Solution**: For **Problem 7**, an orthogonal set of eigenvectors is \( \{(-3, 5, 1), (-3, -2, 1), (1, 0, 3)\} \).

An appropriate unitary matrix is
\[
\begin{pmatrix}
-\frac{3}{\sqrt{35}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{10}} \\
\frac{2}{\sqrt{35}} & \frac{2}{\sqrt{14}} & 0 \\
\frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} & \frac{3}{\sqrt{10}}
\end{pmatrix}.
\]

For **Problem 13**, let \( A \in \mathbb{R}^{n \times n} \) be real skew-symmetric, i.e. \( A^T = -A \). Then \( I \pm A \) are non-singular as otherwise there would be a \( v \neq 0 \) such that \( (I \pm A)v = 0 \iff Av = \mp v \), i.e. \( \mp 1 \) would be eigenvalues of \( A \). However, since \( A \) is real skew-symmetric it has an orthonormal basis of eigenvectors, i.e. one can write \( A = UDU^T \) with \( U \) being a diagonal matrix. Note that the entries of \( D \) and \( U \) do not have to be real. In fact, the condition \( A^T = -A \) implies \( D^T = -D \). In other words, the eigenvalues of \( A \) must be purely imaginary and cannot be \( \mp 1 \). So, \( I \pm A \) are invertible.

Next, we observe that \( (I \pm A)^T = (I \mp A) \) and also
\[
\]

Consequently, we have
\[
((I - A)(I + A)^{-1}) \cdot ((I - A)(I + A)^{-1})^T = (I - A)(I + A)^{-1}(I - A)^{-1}(I + A) = (I - A) \cdot ((I - A)(I + A)^{-1} \cdot (I + A) = (I - A) \cdot ((I + A)(I - A))^{-1} \cdot (I + A).
\]

This being equal to \( I \) is clearly equivalent to \( ((I + A)(I - A))^{-1} = (I - A)^{-1}(I + A)^{-1} \) which is just true. So, \( (I - A)(I + A)^{-1} \) is orthogonal. \[ \square \]

5. (5 pts) Notations as in problem 5 in HW set 5. Let \( V \) be a finite-dimensional vector space over \( \mathbb{R} \), and \( V^* = (V, \mathbb{R}) \). Let \( (\cdot, \cdot) \) denote a (non-degenerate) inner product of \( V \). Prove that

(a) The map \( f : V \to V^* \), defined by \( v \mapsto f(v) = f_v \) where \( f_v(w) = (w, v) \), for all \( v, w \in V \), is an isomorphism (i.e. linear and bijective).

(b) \( B \) is an orthonormal basis of \( V \) if and only if \( f(B) = B^* \) the dual basis of \( V^* \).

(c) A linear transformation \( \phi : V \to V \) is hermitian if and only if \( \phi^* \circ f = f \circ \phi \).
(d) Deduce that if $B$ is an orthonormal basis of $V$ and $A$ the matrix representing $\phi$ with respect to $B$, then $\phi$ is hermitian if and only if $A^T = A$.

Solution. (a) We have
$$f_v(aw_1 + bw_2) = \langle v, aw_1 + bw_2 \rangle = a\langle v, w_1 \rangle + b\langle v, w_2 \rangle = af_v(w_1) + bf_v(w_2)$$
so $f$ is linear. Since $\dim(V) = \dim(V^*)$ to see that $f$ is an isomorphism it suffices to check that $f$ is injective. If $f_v(w) = 0$ for all $w$ then $f_v(v) = 0$ so $\langle v, v \rangle = 0$ so $v = 0$.

(b) Let $B = (v_1, \ldots, v_n)$. $B$ is orthonormal if and only
$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
if and only
$$f_v(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$ The last statement is the definition of $(f(v_1), \ldots, f(v_n))$ being the dual basis to $B$.

(c) We have
$$(0.6) \quad (\phi^* \circ f)(v)(w) = f_v(\phi(w)) = \langle v, \phi(w) \rangle$$
and
$$(0.7) \quad (f \circ \phi)(v)(w) = f_{\phi(v)}(w) = \langle \phi(v), w \rangle.$$ 
$(0.7) = (0.8)$ is the definition of $\phi$ being Hermitian.

(d) This follows from parts (b),(c) and part (a) of Problem 5 on Homework 5. □