1. (6 pts) [Apostol IV.4.6] Let $V$ be the linear space of all real polynomials $p(x)$ of degree $\leq n$. If $p \in V$, define $q = T(p)$ to mean that $q(t) = p(t+1)$ for all real $t$. Prove that $T$ has only the eigenvalue 1. What are the eigenfunctions belonging to this eigenvalue?

Solution. Suppose $\lambda$ is an eigenvalue of $T$, i.e. there exists a $p(t) \neq 0$ s.t. $p(t+1) = \lambda p(t)$. If $a_nT^n$ denotes the highest degree term of $p$ then this is also the highest degree term of $p(t+1)$ (Expand $(t+1)^n$). So, we must have $a_nT^n = \lambda a_nT^n$, i.e. $\lambda = 1$.

Finally, $p(t) = p(t+1)$ means that $p(k)$ is constant for all $k \in \mathbb{Z}$. In particular $p(t) - p(0)$ must have infinitely many zeros on $\mathbb{R}$. Therefore, it must be constant 0 and $p$ is constant $p(0)$. Conversely, any constant polynomial is clearly an eigenvector to the eigenvalue 1. □

2. (7 pts) Consider the matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 2 & 6 \\ 2 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

For each matrix $A = A_1, A_2, A_3$:

- Decide whether $A$ is diagonalizable, i.e., if there exists a diagonal matrix $B$ which is similar to $A$. Justify your answer.
- If it exists, find a diagonal matrix $B$ such that $A$ is similar to $B$. Is $B$ unique? If not, list all such $B$'s.
- If it exists, find an invertible matrix $A$ such that $Q^{-1}AQ$ is diagonal.

Proof. (i) Since $A_1$ is upper triangular, one immediately sees that $\text{char}_{A_1}(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$. Its roots are 1, 2, and 3. Since all its eigenvalues are different, the corresponding eigenvectors span $\mathbb{R}^3$. $A_1$ is diagonalizable.

Observe that $\text{char}_{A_2}(\lambda) = \text{char}_{A_3}(\lambda) = (\lambda - 2)^2(\lambda + 2)$. The eigenvalues of $A_2$ and $A_3$ are 2 and $-2$.

For $A_2$, since multiplicity of the eigenvalue $-2$ is 1, the eigenspace of $-2$ for $A_2$ is dimension 1. To compute the dimension of the the eigenspace of 2 for $A_2$, one solves the homogeneous equation $(2I - A_2) \cdot v = 0$. The solution space is one dimensional. The eigenspace of 2 is one dimensional. $\mathbb{R}^3$ does not have basis of eigenvectors for $A_2$. $A_2$ is not diagonalizable.

The same argument shows that $A_3$ is not diagonalizable.

(ii) Only $A_1$ is diagonalizable. Suppose $\{a_1, a_2, a_3\} = \{1, 2, 3\}$. Then $A_1$ is similar to

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$ 

$A_1$ is not similar to a unique diagonal matrix, since there are 6 possibilities.
(iii) By solving \((I - A_1) \cdot v = 0\), \((2I - A_1) \cdot v = 0\), and \((3I - A_1) \cdot v = 0\), one finds that 1 eigenspace is spanned by \((1, 0, 0)\), the 2 eigenspace is spanned by \((1, 1, 0)\), and the 3 eigenspace is spanned by \((5, 4, 1)\). Therefore, letting

\[
Q = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}
\]

one has

\[
Q^{-1}A_1Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\]

\[\square\]

3. (6 pts) Let \(V = V_3\) be the vector space of degree at most 3 polynomials in one variable \(x\) (with complex coefficients). Let \(T\) be the linear operator

\[T(f) = xf' + f''\]

(a) Calculate the eigenvalues of \(T\).

(b) For each eigenvalue, find a basis for the corresponding eigenspace.

(c) Give a basis of \(V\) for which \(T\) is represented by a diagonal matrix.

Note: Observe that what you have done amounts to finding the values of the parameter \(\lambda\) for which the differential equation \(\lambda f - xf' - f'' = 0\) has a polynomial solution, and find the solutions for such values.

Solution. We first calculate the matrix corresponding to \(T\) with respect to the basis \(\{1, x, x^2, x^3\}\):

\[
A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}
\]

This matrix is upper-triangular, so its diagonal entries must be the eigenvalues. Therefore, the eigenvalues are 0, 1, 2, 3.

We can calculate the \(\lambda\)-eigenspaces by finding the solutions \(v\) to \((A - \lambda I)v = 0\). We have

- 0-eigenspace = \(\text{span}\{(1, 0, 0, 0)^T\}\)
- 1-eigenspace = \(\text{span}\{(0, 1, 0, 0)^T\}\)
- 2-eigenspace = \(\text{span}\{(1, 0, 1, 0)^T\}\)
- 3-eigenspace = \(\text{span}\{(0, 3, 0, 1)^T\}\).

These vectors form a basis for \(V\), and using this eigenbasis, we can represent \(T\) as a diagonal matrix. For example, if

\[
B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
is the base change matrix to the eigenbasis, we have
\[
B^{-1}AB = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 0 & 2 & 0 \\
0 & 1 & 0 & 6 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix} \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

\[
= \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 0 & 2 & 0 \\
0 & 1 & 0 & 9 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}.
\]

\[
= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}.
\]

\[\square\]

4. (6 pts) A population of fish is divided into red, blue and yellow. It has been observed that 1/4 of the descendants of red fish are red, 1/2 are blue and 1/4 yellow; descendants of blue fish are 1/3 red, 1/3 blue, and 1/3 yellow; descendants of yellow fish are 1/3 red, 1/9 blue, and 5/9 yellow. We record this data in the following probability matrix
\[
P = \begin{bmatrix}
1/4 & 1/3 & 1/3 \\
1/2 & 1/3 & 1/9 \\
1/4 & 1/3 & 5/9
\end{bmatrix}.
\]
Assume the current population \( u_0 \) is 1/3 red, 1/3 blue, and 1/3 yellow.

(a) Compute (as it exists!) a population \( v_1 \) which remains unchanged (stable) as generations pass.

(b) Compute all eigenvalues and eigenvectors of the matrix \( P \).

(c) Compute a closed formula for the \( n \)-th generation \( u_n \).

(d) Prove that with the passing of generations, the fish population approaches \( v_1 \) (i.e. \( \lim_{n \to \infty} u_n = v_1 \)).

Proof. (a) If such a population \( v_1 \) exists, it should satisfy the following equation:
\[
P v_1 = v_1.
\]
Solve the equation we have \( v_1 = \lambda \begin{pmatrix} 28 \\ 27 \\ 36 \end{pmatrix} \). After normalizing it we get
\[
v_1 = \left( \frac{4}{13}, \frac{27}{91}, \frac{36}{91} \right)^T.
\]

(b) Let \( \det(P - \lambda I) = 0 \), we have
\[
\det \begin{pmatrix}
\frac{1}{4} - \lambda & \frac{1}{3} & \frac{1}{9} \\
\frac{1}{4} & \frac{1}{3} - \lambda & \frac{1}{3} \\
\frac{1}{4} & \frac{1}{3} & \frac{2}{5} - \lambda
\end{pmatrix} = 0.
\]
We simplify the equation and obtain:
\[
-\lambda^3 + \frac{41}{36} \lambda^2 - \frac{13}{108} \lambda - \frac{1}{54} = 0.
\]
By part (a), we know that $\lambda = 1$ is an eigenvalue, so we can rewrite the above equation as

$$-(\lambda - 1)(\lambda - \frac{2}{9})(\lambda + \frac{1}{12}) = 0.$$ 

Hence $P$ has three (distinct) eigenvalues: $\lambda = 1, \frac{2}{9}, -\frac{1}{12}$. Since the eigenvalues are distinct, so we can find an one-dimensional eigenspace for each eigenvalue.

When $\lambda = 1$, by (a) we have an eigenvector $(\frac{4}{13}, \frac{27}{91}, \frac{36}{91})^T$.

When $\lambda = \frac{2}{9}$, the corresponding eigenvector is (the multiple of) $(0, 1, -1)^T$.

When $\lambda = -\frac{1}{12}$, the corresponding eigenvector is (the multiple of) $(11, -\frac{14}{3}, 3)^T$.

(c) By definition we have $u_n = P^n u_0$, where $u_0 = (1/3, 1/3, 1/3)^T$. Since $P$ has three distinct eigenvalues so $P$ is diagonalizable. We choose $Q = \left( \begin{array}{ccc}
\frac{4}{13} & 0 & 11 \\
\frac{27}{91} & 1 & -14 \\
\frac{36}{91} & 1 & 3 
\end{array} \right)$,

then we have that $Q^{-1}PQ = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & \frac{2}{9} & 0 \\
0 & 0 & -\frac{1}{12}
\end{array} \right)$. So $P^n = Q \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & \left(\frac{2}{9}\right)^n & 0 \\
0 & 0 & \left(-\frac{1}{12}\right)^n
\end{array} \right)Q^{-1}$,

hence we have that

$$u_n = Q \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & \left(\frac{2}{9}\right)^n & 0 \\
0 & 0 & \left(-\frac{1}{12}\right)^n
\end{array} \right)Q^{-1}u_0.$$ 

We first compute the inverse of $Q$ and have

$$Q^{-1} = \left( \begin{array}{ccc}
\frac{45}{9} & \frac{24}{71} & -\frac{53}{7} \\
\frac{27}{91} & \frac{77}{4} & -\frac{231}{7} \\
\frac{36}{91} & -\frac{53}{7} & \frac{143}{7}
\end{array} \right).$$ 

Finally we get the closed formula of $u_n$:

$$u_n = \left( \frac{4}{13} + \frac{1}{39}\left(-\frac{1}{12}\right)^n, \frac{27}{91} + \frac{16}{231}\left(\frac{2}{9}\right)^n - \frac{14}{429}\left(-\frac{1}{12}\right)^n, \frac{36}{91} + \frac{16}{231}\left(\frac{2}{9}\right)^n + \frac{1}{143}\left(-\frac{1}{12}\right)^n \right)^T.$$ 

(d) When $n$ goes to infinity, both $\left(\frac{2}{9}\right)^n$ and $\left(-\frac{1}{12}\right)^n$ converge to 0. Hence the closed formula of $u_n$ tells us that $u_n$ will converge to $\left( \frac{4}{13}, \frac{27}{91}, \frac{36}{91} \right)^T$. In other words,

$$\lim_{n \to \infty} u_n = v_1.$$ 

□