Math 1b Analytical – Homework Set 5  
Due 4:00 pm on TUESDAY, February 16

Read from the textbook: Chapter 3, Sections 1–17 (and start reading Chapter 4).

You can collaborate on the problems as long as you write up all solutions in your own words and understand those solutions.

1. (10 pts) From Ch. 3.17 in Apostol: Problems 7 and 8

2. (5 pts) Let $M$ be a $n \times n$-matrix, written in block form:

$$
M = \begin{pmatrix}
A & D & E \\
0 & B & 0 \\
0 & F & C \\
\end{pmatrix}
$$

where $A, B, C$ are square matrices. Prove that $\det(M) = \det(A) \det(B) \det(C)$. Note that $D, E,$ and $F$ need not be square. (Hint: first prove a formula for the determinant of matrices that look the lower right hand corner of $M$.)

3. (5 pts) From Ch. 4.4 in Apostol: Problems 1 and 2.

4. (5pts) Let $V$ be a vector space over $\mathbb{R}$, dim $V = n$, and $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of $V$.

   Let $V^* = \mathcal{L}(V, \mathbb{R})$ be the space of linear functions from $V$ to $\mathbb{R}$. For each $i$, let $f_i$ be the linear function defined by $f_i(v_j) = 1$ if $i = j$ and 0 otherwise. Recall that $V^*$ is a vector space and $\mathcal{B}^* = \{f_1, \ldots, f_n\}$ is a basis of $V^*$. We call $V^*$ the dual space of $V$, and $\mathcal{B}^*$ the basis of $V^*$ dual to $\mathcal{B}$.

   Let $W$ be another vector space over $\mathbb{R}$, dim $W = m$, and basis $\mathcal{C}$. Let $W^*$ be the dual space and $\mathcal{C}^*$ the dual basis. Let $\phi : V \to W$ be a linear transformation. For $f \in W^*$ define $\phi^*(f) = f \circ \phi \in V^*$. Recall that $\phi^* : W^* \to V^*$ is a linear transformation.

   • Prove that if $A$ is the matrix associated to $\phi$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, then the matrix of $\phi^*$ with respect to the bases $\mathcal{C}^*$ and $\mathcal{B}^*$ is $A^T$, the transpose of $A$.
   • Deduce that $(AB)^T = B^T A^T$. 
