Let \( f(x) \) be a function whose domain contains an interval of the form \([a, +\infty)\) for some constant \( a \). We say that \( \lim_{x \to \infty} f(x) = L \), if for every positive \( \epsilon \) there is a number \( M \) such that \( |f(x) - L| < \epsilon \) for all \( x > M \). We say that \( \lim_{x \to \infty} f(x) = \infty \), if for every positive \( N \) there is a number \( M \) such that \( f(x) > N \) for all \( x > M \).

**Problem 1.** Using the above definitions, compute the following limits or show they don’t exist.

1. \( \lim_{x \to \infty} \frac{1 + x^2}{x^2} \)
2. \( \lim_{x \to \infty} x(1 + \sin^2(x)) \)
3. \( \lim_{x \to \infty} x \sin^2(x) \)
4. \( \lim_{x \to \infty} x(\sqrt{x + 2} - \sqrt{x}) \)

**Problem 2.**

1. Let \( e \) be the positive real number such that \( \ln e = 1 \). Show that the function \( f(x) = \frac{\ln x}{x} \) is strictly decreasing for \( x > e \).
2. Using the definition \( \ln x = \int_1^x \frac{dt}{t} \), show that for every integer \( n > 1 \):

   \[
   \sum_{k=2}^{n} \frac{1}{k} < \ln n < \sum_{k=1}^{n-1} \frac{1}{k}.
   \]

3. For any \( m \geq 1 \), show that:

   \[
   \sum_{k=1}^{2^m-1} \frac{1}{k} \leq m.
   \]

4. Conclude that \( \lim_{x \to \infty} \frac{\ln x}{x} = 0 \).

**Problem 3.**

1. Show that \( e^x > 1 + x \) for all \( x > 0 \). (Hint: Show that the function \( f(x) = e^x - 1 - x \) is strictly increasing for \( x > 0 \))
2. For any \( n \geq 1 \), show that \( e^x > 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \). (Hint: Start with the inequality of the previous part and integrate)
3. Conclude that, for any \( n \geq 1 \), \( \lim_{x \to \infty} \frac{e^x}{x^n} = \infty \).

**Problem 4.** Find all functions \( f \), defined and continuous on the positive real axis, that satisfy:

\[
\int_1^{xy} f(t) dt = y \int_1^x f(t) dt + x \int_1^y f(t) dt
\]

for all \( x, y > 0 \).
Problem 5. (1) Let $h$ be continuous and $f$ and $g$ differentiable and consider the function:

$$F(x) = \int_{f(x)}^{g(x)} h(t) dt$$

Show that

$$F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$$

(2) Find the derivatives of each of the following functions:

$$F(x) = \int_3^{f(x) \sin^3 t \, dt} \frac{1}{1 + \sin^6 t + t^2} dt$$

$$G(x) = \int_{15}^{x} \left( \int_8^{y} \frac{1}{1 + t^2 + \sin^2 t} dt \right) dy$$