Problem 1. Find the value of the following limits. In each case, use the $\epsilon$-$\delta$ definition to prove that your proposed value is indeed the limit.

1. \( \lim_{x \to 1} \frac{x^2 - 1}{x + 1} \)
2. \( \lim_{x \to y} \frac{x^n - y^n}{x - y} \), where \( n \in \mathbb{N} \) is fixed.
3. \( \lim_{h \to 0} \frac{\sqrt{a + h} - \sqrt{a}}{h} \), where \( a > 0 \) is fixed.
4. \( \lim_{x \to 0} \frac{\sqrt{x \cos(x)}}{1 - \sin^2(x)} \)

Problem 2. We define \( \lim_{x \to x_0} f(x) = \infty \) to mean that for all \( N \in \mathbb{N} \) there is a \( \delta > 0 \) such that \( |f(x)| > N \) whenever \( x \) satisfies \( 0 < |x - x_0| < \delta \). Show that

1. \( \lim_{x \to 3} \frac{1}{(x-3)^2} = \infty \).
2. Prove that if for some \( c > 0 \), \( |f(x)| > c \) for all \( x \), and \( \lim_{x \to x_0} g(x) = 0 \), then

\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = \infty \]


Problem 4. (1) Prove that given two real numbers \( a, b \), the following formulas hold true:

\[ \max\{a, b\} = \frac{1}{2} [(a + b) + |a - b|] , \]
\[ \min\{a, b\} = \frac{1}{2} [(a + b) - |a - b|] . \]

(2) Given two continuous functions \( f, g \), use the previous part to show that the functions \( \max\{f, g\} \) and \( \min\{f, g\} \) are continuous.

Problem 5. Suppose a function \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(x + y) = f(x) + f(y) \), all \( x, y \in \mathbb{R} \), and that \( f \) is continuous at 0. Prove that \( f \) is continuous on all of \( \mathbb{R} \).

Problem 6. Suppose that \( f \) and \( g \) are continuous functions on \( \mathbb{R} \), that \( f^2 = g^2 \), and that \( f(x) \neq 0 \) for all \( x \). Prove that either \( f(x) = g(x) \) for all \( x \), or else \( f(x) = -g(x) \) for all \( x \).

Problem 7. Apostol, Exercises 3.6, Page 139: Number 27.