The homework set is due Thursday 1/22 at 2pm.

1. Let $\alpha : (a,b) \to \mathbb{R}^3$ be a curve parametrized by arc length with positive curvature $\kappa(s)$ for all $s \in (a,b)$. Let $P$ be a plane satisfying:
   a. $P$ contains the tangent line at $s$.
   
   b. Given any neighborhood $J \subset (a,b)$ of $s$, there exist points of $\alpha(J)$ in both sides of $P$.

Prove that $P$ is the osculating plane of $P$ at $s$.

2. Let $V = -yU_1 + xU_3$ and $W = \cos x U_1 + \sin x U_2$. Express the following covariant derivatives in terms of $U_1, U_2$ and $U_3$:
   (a) $\nabla_W V$.
   (b) $\nabla_V (xV - zW)$.
   (c) $\nabla_V (\nabla_V W)$.

3. For a Frenet curve $\beta$ in $\mathbb{R}^n$, prove that the Frenet curvatures and the Frenet n-frame are invariant under Euclidean motions. (A Euclidean motion is a map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $B$ can be written as $B(x) = Ax + b$ where $A^{-1} = A^T$ and $\det(A) = 1$.)

4. Let $X$ be the special vector field $\Sigma x_i U_i$, where $x_1, x_2$ and $x_3$ are the natural coordinate functions of $\mathbb{R}^3$. Prove that $\nabla_V X = V$ for every vector field $V$.

5. Let $\beta$ be a regular smooth curve in $\mathbb{R}^3$ that is parametrized by arc length and whose image lies on the unit sphere $S^2 \subset \mathbb{R}^3$. Set $J := Det(\beta, \beta', \beta'')$. Prove that $\kappa = \sqrt{1 + J^2}$ and $\tau = J'/(1 + J^2)$.

6. Find the connection forms of the natural frame field $U_1, U_2, U_3$. 