Math 108b Problem set 7 due: Wednesday March 11, 2015 4:00 P.M

Do problem 1 from Chapter 7 of Wheeden-Zygmund

1. Let $f$ be a real valued function on $[0,1]$ whose derivative exists and is continuous at every point of $[0,1]$. Then the graph of $f$

$$G = \{(x, f(x)) : x \in [0,1]\}$$

is a rectifiable curve. Suppose that the length of $G$ is equal to $M$. Let $K > 0$ be a real number. Show that there is a measurable set $E \subset [0,1]$ with

$$|E| \leq \frac{10M}{K},$$

so that for any $x, y \in [0,1]$, with neither $x$ nor $y$ in $E$, we have the estimate:

$$|f(x) - f(y)| \leq K|x - y|.$$

Hint: Observe that the length of $G$ gives a bound on the integral of the absolute value of the derivative of $f$. Then apply the Hardy-Littlewood estimate Lemma 7.9, for which our proof gives the constant 5 when $n = 1$. Thanks to Alex Stark for pointing out to me that in fact the constant is around 1.58.

2. Let $E \subset [0,1]$ be a measurable set with $|E| = \epsilon > 0$. Let

$$f(x) = \chi_E(x).$$

Define the minimal function

$$(m_1f)(x) = \inf_{r<1} \frac{1}{r} \int_{I_{r}(x)} |f(x)| dx,$$

where $I_r(x)$ is the interval centered around $x$ with length $r$. (This differs from the Hardy Littlewood maximal function in that the sup becomes an inf and the intervals are always chosen with length less than 1.) Define

$$E_{bad} = \{x \in E : M_1 f(x) < \frac{\epsilon}{100}\}.$$

Show that

$$|E_{bad}| \leq \frac{\epsilon}{10}.$$

(In other words most points of $E$ are good.) Hint: For every point $x \in E_{bad}$, there is $r_x$ so that $|I_{r_x}(x) \cap E| \leq \frac{\epsilon}{100} r_x$. Now apply the ideas in the proof of the simple Vitali lemma, Lemma 7.4 to the collection of intervals $I_{r_x}(x)$. Estimate the sum of the lengths of the intervals you have selected and the total measure of the part of $E$ inside their quintuples.