Ma 145a: Homework set 5, Due December 10 at noon

Choose three from the following five problems to turn in. In this set we discuss some fact regarding $k$-representations of a finite group $G$ while $k$ has positive characteristic.

1. Find $R_k(\mathbb{Z}_3)$ and $R_k(A_4)$.

2. Let $R$ be a discrete valuation ring (local commutative principal ideal domain which is not a field) with fraction field $K$ of characteristic zero. Assume $\pi_R$ is a uniformizer (i.e. it generates the unique maximal ideal of $R$) and $k = R/\pi_R R$ is its residue field. Let $G$ be a finite group. Show that

(a) any finitely generated $K[G]$-module $V$ contains a lattice which is $G$-invariant.

(b) the image of the $k[G]$-module $L/\pi_R L$ in $R_k(G)$ is unique for $L$ any such $G$-invariant lattice. (This is the same as saying the composition factors are unique up to isomorphism.)

(A lattice $L$ of $V$ is a free $R$-submodule of $V$ such that a $K$-basis of $V$ generates $L$.)

Some counterexamples: Take $G$ to be the $2$-group $\{1, \sigma\}$ of order $2$ and take $R$ to be the localization $\mathbb{Z}(2)$ of $\mathbb{Z}$ at the prime ideal $(2)$. Then $K = \mathbb{Q}$ and $k = \mathbb{F}_2$. Consider $V = \mathbb{Q}e_1 + \mathbb{Q}e_2$ with $\sigma e_1 = e_2$.

Take $L_1 = R(e_1 + e_2) + R(e_1 - e_2)$ and $L_2 = Re_1 + Re_2$. We have $2L_1 \subset L_2 \subset L_1$ but $L_1 = R(e_1 + e_2) \oplus R(e_1 - e_2)$ is decomposable while $L_2$ is indecomposable but contains the $R[G]$-module $R(e_1 + e_2)$ and is not simple. ($L_1 \not\simeq L_2$ as $R[G]$-modules) We have $[L_1] = [L_2] = [R(e_1 + e_2)] + [R(e_1 - e_2)]$. In this case, $R_k(G) \simeq \mathbb{Z}$ and for all $R$-module $L$ of finite length, $[L/2L] \mapsto \dim_k L/2L$.

3. Assume $G$ is a $p$-group with order $p^r$ and $k$ is an algebraically closed field of characteristic $p$. Classify all simple $k[G]$-modules.

4. Assume $F$ is an arbitrary field and $G$ is a finite group. Let $V$ be a simple $F[G]$-module. Let $N$ be a normal subgroup of $G$ and $W$ a simple $F[N]$-submodule of $V$ and let $K = \text{Norm}_G(W,N)$ be the subgroup $\{g \in G \mid gW \simeq W\}$. Write $W$ for the $W$-isotypic part of $V$. Show that $W$ is a simple $F[K]$-module and $V \simeq \oplus_{g \in G/K} gW$, i.e. $V \simeq \text{Ind}_K^G W$.

5. Let $R$ be a commutative ring, and let $P$ be a finitely generated $R[G]$-module which is projective over $R$. Show that $P$ is a projective $R[G]$-module if and only if for all maximal ideal $m$ of $R$, $P/mP$ is a projective $(R/m)[G]$-module. (A left $A$-module $M$ is projective if the functor $\text{Hom}_A(M, -)$ is exact.)