Choose three from the following four problems to turn in.

1. Recall that the Littlewood-Richardson numbers $N_{\lambda\mu\nu}$ appear in the coefficients of the expansion $S_\lambda S_\mu = \sum_{\nu} N_{\lambda\mu\nu} S_\nu$ and equal to the multiplicity

$$N_{\lambda\mu\nu} = \dim \text{Hom}_{S_\lambda \times S_\mu}(V_\lambda \boxtimes V_\mu, V_\nu).$$

Show the following identity on Weyl’s modules

$$S_\nu(V \oplus W) = \bigoplus_{\lambda,\mu} N_{\lambda\mu\nu}(S_\lambda V \otimes S_\mu W).$$

2. Use the previous result to deduce the Branching law for $GL_n(\mathbb{C})$:

$$\text{Res}_{GL_n(\mathbb{C})}^{GL_{n+1}(\mathbb{C})} S_\nu(\mathbb{C}^{n+1}) = \bigoplus_{\lambda \geq \nu_1 \geq \cdots \geq \nu_{n+1}} S_\lambda(\mathbb{C}^n)$$

by embedding $GL_n$ into $GL_{n+1}$ via $GL_n \hookrightarrow GL_n \times GL_1 \hookrightarrow GL_{n+1}$.

3. In the lecture we introduced a pairing $\langle \cdot, \cdot \rangle$ on the symmetric polynomials defined by assigning $\langle H_\lambda, M_\mu \rangle = \delta_{\lambda\mu}$. Under this pairing $\langle S_\lambda, S_\mu \rangle = \delta_{\lambda\mu}$ and the Schur polynomials are dual and give an orthonormal basis. The ring of symmetric polynomials is a graded ring with $H_m$ and $E_m$ of degree $m$. Consider the graded ring

$$\Lambda = \mathbb{C}[H_1, H_2, \ldots] = \mathbb{C}[E_1, E_2, \ldots]$$

and define a ring homomorphism $\partial : \Lambda \to \Lambda$ by

$$\partial(H_m) = E_m.$$

Then a fact is $\partial(S_\lambda) = S_{\lambda'}$ where $\lambda'$ denotes the conjugate partition of $\lambda$. (This is by the identities $S_\lambda = [H_{\lambda'+j-i}] = [E_{\lambda'+j-i}]$ which are known as the Giambelli’s formula from geometry.)

(a) Show that $\partial$ is an involution $\partial^2 = \partial$.

(b) Show that $E_{\lambda'} = \sum_{\mu} K_{\mu'\lambda'} S_\mu$. (This is a dual statement to $H_\lambda = \sum_{\mu} K_{\mu\lambda} S_\mu$.)

(c) Show that

- $\text{Sym}^{\lambda_1} V \otimes \text{Sym}^{\lambda_2} V \otimes \cdots \otimes \text{Sym}^{\lambda_k} V \cong \bigoplus_{\mu} K_{\mu\lambda} S_\mu V$

- $\Lambda^{\lambda_1} V \otimes \Lambda^{\lambda_2} V \otimes \cdots \otimes \Lambda^{\lambda_k} V \cong \bigoplus_{\mu} K_{\mu'\lambda'} S_\mu V$.

(d) Deduce that $V_\lambda$ is the only irreducible representation of $S_n$ that occurs in both $\mathbb{C}[S_n]a_\lambda$ and $\mathbb{C}[S_n]b_\lambda$ and it occurs with multiplicity 1.

4. The problem is aimed to prove that each $f \in R(G)$ such that $f(1) = 0$ is a $\mathbb{Z}$-linear combination of the element of the form $\text{Ind}_E^G(\alpha - 1)$, where $E$ is an elementary subgroup of $G$ and $\alpha$ is a character of degree 1.

(a) Let $R'_0(G)$ be a subgroup of $R(G)$ generated by the $\text{Ind}_E^G(\alpha - 1)$s, and let $R'(G) = \mathbb{Z} + R'_0(G)$. Show that if $H$ is a subgroup of $G$, $\text{Ind}_H^G$ maps $R'_0(H)$ into $R'_0(G)$.  

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(b) Suppose that $H$ is normal in $G$ and that $G/H$ is abelian. Show that $\text{Ind}^G_H$ maps $R'(H)$ into $R'(G)$. (It is enough to show that $\text{Ind}^G_H 1$ belongs to $R'(G)$. Then this follows that $\text{Ind}^G_H 1$ is the sum of $(G : H)$ characters of degree 1 whose kernel contains $H$.)

(c) Suppose $G$ is elementary (so nilpotent). Let $Y$ be the set of all maximal subgroups of $G$. Notice that all maximal subgroups of an elementary group are normal of prime index. Deduce that $R(G)$ is generated by characters of degree 1 and $\bigoplus_{H \in Y} \text{Ind}^G_H R(H)$. Then use the previous results to show that $R(G)' = R(G)$ by applying induction on $|G|$.

(d) Use Brauer’s Theorem to show that for general finite group $G$, $R(G)' = R(G)$. 
