1. Let \((\rho, V)\) be a representation of \(G\) of degree \(n\). Assume \(\det(\rho(g)) = 1\) for all \(g \in G\). Then for \(0 \leq k \leq n\), the representation \(\wedge^k V\) is isomorphic to \(\wedge^{n-k} V^*\).

2. Let \((\rho, V)\) be an irreducible representation of degree \(n\) of a finite group \(G\). Let \(\chi_V\) be its character. Assume \(Z\) is the center of \(G\). Let \(g\) and \(c\) be the order of \(G\) and \(Z\) respectively.
   
   (a) Show that \(Z\) acts on \(V\) by scalars via \(\rho|_Z\). Deduce from it that \(|\chi_V(z)| = n\).
   
   (b) Prove that \(n^2 \leq g/c\). (We have seen \(n \leq g/c\).)
   
   (c) Show that if \(\rho\) is faithful (i.e. \(\rho : G \to \text{GL}(V)\) is injective) then \(Z\) is a cyclic group.

3. Let \(H\) be a subgroup of a finite group \(G\). Show that each irreducible representation of \(G\) is contained in a representation induced by an irreducible representation of \(H\).

4. Assume \(K\) has positive characteristic \(p\) and \(G\) has order \(g\). Show that the following two properties are equivalent:
   
   (a) The group algebra \(K[G]\) is semisimple. (Equivalently, every \(K[G]\)-module is semisimple.)
   
   (b) The characteristic \(p\) of \(K\) does not divide the order \(g\) of \(G\).