Lecture 9: Branching into algebraic topology

There are some natural connections between flows on graphs with cohomology. Firstly, this is because the space of flows on a graph is capturing the topology of the graph. That is to admit a certain number of flows requires that one has holes. We wish to make this precise by analysing mechanisms for computing the holes in a space. Graphs are often the most basic examples used in algebraic topology.

Secondly may expand on the point made in the book where it says that circulations on a digraph are called 1-cycles in algebraic topology. We will compute the homology via some of the tools of algebraic topology, namely the Hurewucz theorem. A nice and more comprehensive introduction to some of the concepts here may be found in “Algebraic Topology” by Allen Hatcher, which is freely available at http://www.math.cornell.edu/~hatcher/AT/AT.pdf

Let us start by fixing some notation

1. Homotopy

Our first construction is the fundamental group, defined by Poincare. The idea is that we generalize the notion of a path. By introducing an equivalence on paths we form a group.

Given a topological space, \(X\), we can redefine a path as a continuous function \(\gamma : [0, 1] \to X\).

A path is called closed at some point \(x_0\) if

\[\gamma(0) = \gamma(1) = x_0.\]

Given two closed paths at \(x_0\), \(\gamma_0\) and \(\gamma_1\), we can define the product

\[\gamma_0 \cdot \gamma_1 = \begin{cases} 
\gamma_0(2t) & \text{if } 0 \leq t \leq 1/2, \\
\gamma_1(2t - 1) & \text{if } 1/2 \leq t \leq 1.
\end{cases}\]

We define an equivalence, \(\gamma_0 \sim \gamma_1\) if we may continuously deform the path of \(\gamma_0\) to the path of \(\gamma_1\).

**Definition 1.1.** The fundamental group of \(X\) at the point \(x_0\) is a group consisting of all equivalence classes of close pathed paths at \(x_0\). It is denoted,

\[\pi_1(X, x_0) = \{[\gamma] : [0, 1] \to X, \gamma(0) = \gamma(1) = x_0\}.\]
One of the original contexts in which this arises is the monodromy of a complex function with singularities.

Given a path from $x_0$ to some $x_1$,
\[ h : [0, 1] \to X \]
where $h(0) = x_0$ and $h(1) = x_1$, we can define the map
\[ \beta_h : \pi_1(X, x_0) \to \pi(X, x_1) \]
by sending $f \to h \circ f \circ h^{-1}$. This gives us the following result.

**Proposition 1.2.** If $X$ is path connected, then $\pi_1(X, x_0) \cong \pi(X, x_1)$ for any $x_1$.

The fundamental group at a point, $x_0 \in X$, depends on the path-connected component of $X$. We will assume our space is path-connected.

Let $G$ be a graph, we can endow $G$ with a topology by letting each edge be a closed line segment with the usual topology. The two examples below are a path and a tree.

A polygon $P_n$. $\pi_1(P_n, v) \cong \mathbb{Z}$. A tree $T$. $\pi_1(T, v) \cong \{0\}$.

These examples are simple, but illustrate two basic principles, that given any tree, the group is trivial and any loop gives a single generator. In general, we do not expect the generators given by a loop to commute. For example, if we take a
graph equivalent to a figure 8, such as the graph below, we see that there is no way to continuously deform a path around each loop in a way that swaps them.

\[ \pi_1(G, x_0) = \langle a, b \rangle = \text{which is the free group on two symbols.} \]

More generally, if you fix a spanning tree, there is a unique path (up to equivalence) from any given vertex to the endpoints of any edge not in the tree. Each edge not in the tree gives rise to a copy of \( P_n \) inside the graph. This allows us to completely characterize the homotopy group of a connected graph.

**Theorem 1.3.** Given a graph, \( G \), with a spanning tree \( T \), the fundamental group of the graph is the free group generated by the edges in \( E(G) \setminus E(T) \).

Another useful characterization is in terms of the Euler characteristic,

\[ \chi(G) = |V(G)| - |E(G)| \]

in which any tree has Euler characteristic \( \chi(G) = +1 \). For more general graphs, \( \pi_1(G) \) is the free group on \( 1 - \chi(G) \) generators.

### 2. Graph homology

While I would like to jump to cohomology, as it relates to flows, one doesn’t do justice to the theory without introducing (simplicial homology). The homology groups are generally much easier to compute than the homotopy groups, and consequently one usually will have an easier time working with homology to aid in the classification of spaces. We will find for graphs, the homology groups may be computed using a well known extension of the results of Poincare.

We start by defining what we mean by a chain complex, or CW complex. Let us first define a arbitrary CW complex inductively,

1. We start with a discrete set, \( X^0 \), whose points are regarded as 0-cells.
2. Inductively, form the \( n \)-skeleton, \( X^n \) from \( X^{n-1} \) by attaching \( n \)-cells \( e_n^\alpha \) via maps

\[ \varphi_\alpha : S^{n-1} \to X^{n-1} \]

which assigns the boundary of the cell to some copy of \( S^{n-1} \) in the \( n - 1 \) skeleton.
3. We stop this inductive process at a finite stage setting \( X = X^n \) (although infinite extensions are possible).

Let \( C_n \) be the free abelian group with a basis consisting of all \( n \)-simplicies. That is we have a formal \( \mathbb{Z} \)-linear combinations of the \( n \)-cells. Each element of \( C_n \)
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is called a chain, 

$$x = \sum_{e_i} n_i [e_i^n]$$

where we have used the notation $[e_i^n]$ to remind us that this is a formal sum. For the purposes of homology, we will number the vertices, $v_0, \ldots, v_n$. This introduces an orientation, which determines the boundary map, $\delta : C_n \to C_{n-1}$. If the simplex under this ordering is $e = [v_0, v_1, \ldots, v_n]$, then the boundary map

$$\delta : [v_0, \ldots, v_n] \to \sum (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n],$$

which is depicted below.

In this way, the boundary map is considered a map

$$C_n \xrightarrow{\delta} \cdots \xrightarrow{\delta} C_1 \xrightarrow{\delta} C_0 \xrightarrow{\delta} 0$$

where the most important relation is given by

$$\delta \circ \delta = 0,$$

hence, the Im $\delta \subseteq$ Ker $\delta$. The $n$-th (simplicial) homology is the quotient

$$H_k(X, \mathbb{Z}) = \text{Ker} \delta / \text{Im} \delta$$

which we may compute for specific examples. In this way, a graph may be identified as a 1-dimensional cell complex $X = X^1$. It consists of vertices (the 0-cells) to which edges (the 1-cells) are attached, where $C_2$ is necessarily trivial, so we have

$$0 \xrightarrow{\delta} C_1 \xrightarrow{\delta} C_0 \xrightarrow{\delta} 0.$$ 

In the example of a tree, we may argue inductively. Suppose

$$x = \sum n_i e_i \in \text{Ker} \delta$$

then if $v_i$ is vertex of degree 1, incident to $e_j = \{v_i, v_k\}$, then we apply $\delta$,

$$\delta x = \delta e_j + \sum_{i \neq j} \delta e_i = n_j [v_i] + \text{terms not involving } v_i,$$

hence $n_j$ is 0. Inductively, any element of the kernel only depends on the subgraph with $e_j$ removed, hence, $H_1(G, \mathbb{Z}) = 0$. We have just one path connected component, $H_0(X, \mathbb{Z}) = \mathbb{Z}$. 

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\[ \text{Diagram:} \]

\[ \begin{aligned} 
\delta[v_0, v_1] &\to [v_1] - [v_0] \\
\delta[v_0, v_1, v_2] &\to [v_1, v_2] \\
\delta[v_0, v_1, v_2] &\to [v_1, v_2, v_3] - [v_0, v_2, v_3] \\
- [v_0, v_2] &+ [v_0, v_1] \\
+ [v_0, v_1, v_2] &- [v_0, v_1, v_2] 
\end{aligned} \]
For a polygon, the image of each edge is $\delta[e_i] = [v_{i+1}] - [v_i]$ which tells us $\delta \sum e_i = 0$, since the image of $\delta: C_2 \to C_1$ is 0, then $H_1(P_n, \mathbb{Z}) = \mathbb{Z}$. Similarly, $H_0(P_n, \mathbb{Z}) = \mathbb{Z}$. For a figure 8, both edges are part of the kernel, hence, $H_1(8, \mathbb{Z}) = \mathbb{Z}^2$ and $H_0(8, \mathbb{Z}) = \mathbb{Z}$.

For an arbitrary graph, we invoke the following theorem, called the Hurewicz theorem, which states that the first non-zero homotopy group is related to the first homology group via the Hurewicz homomorphism

$$H_1(G, \mathbb{Z}) \cong \pi_1(G)/[\pi_1(G), \pi_1(G)],$$

which is the abelianization of the fundamental group. I.e., this is a free abelian group generated by the edges not in a spanning tree.

### 3. Cohomology of graphs

The idea of cohomology is dual to the idea of homology. Given a complex, $X$, with $k$-skeletons, $X^k$, $k \leq n$, we consider the ring of functions on the $n$-cells

$$C^k(X, H) = \{f : X^k \to H\}.$$

Analoguously to the homological setting, we have an operator call the coboundary $\delta^*$, whose application to the an element of $C^k$ gives an element of $C^{k+1}$, specified by

$$(\delta^* f)([v_0, \ldots, v_k]) = \sum (-1)^i f([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k]).$$
This satisfies $\delta^* \circ \delta^* = 0$ for much the same reasons, hence, $\text{Im} \delta \subseteq \text{Ker} \delta$. This gives us a sequence

$$0 \xrightarrow{\delta} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \ldots$$

where the cohomology is given by

$$H^k(X, H) = \text{Ker} \delta / \text{Im} \delta,$$

In the simplest case, for a graph, we have two rings of interest

$$C^0 = \{ \phi : V(G) \to H \},$$
$$C^1 = \{ \phi : E(G) \to H \}.$$

The if $f : V(G) \to H$ is any function in the kernel, then

$$(\delta f)[v_0, v_1] = f([v_1]) - f(v_2) = 0,$$
which implies $f(v_1) = f(v_0)$

in particular, for a connected graph, $f(v_i) = f(v_j)$, for each $v_j$, hence, since the image is trivial, $H^0(G, H) = H$. In particular, it holds for any of the examples we had before. The first cohomology group is $H^1(G, H) = C^1(X, H) / \text{Im} \delta$. In particular, if $f, g \in C^0(X, H)$ are considered homologous if

$$f - g = \delta h$$

for some $h \in C^0(X, H)$.

For general graphs, once again, we have the result that the cohomology is a copy of $H$ for each edge not in a spanning tree.

To see the relation between this and the last lecture, we let $H = \mathbb{R}$, then a flow is a 1-chain. The conservation condition may be expressed in terms of the boundary map, $\delta : C^1 \to C^0$, where

$$\delta \phi(v) = \sum_{e \text{ into } v} \phi(v) - \sum_{e \text{ out of } v} \phi(v).$$

Using this boundary map, the condition is written

$$\delta f = 0.$$

For planar graphs, we can attach faces, in which a boundary circulation is defined as the boundary of a 2-chain.