Homework 4 : Ma121a - Combinatorics

Homework is due on Thursday the 20th of November at 12:00. While collaboration is encouraged, you must write your own solutions.

1) Determine
\[ \sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor \]  
(Problem 10C) (3 marks)

2) Count the number of permutations \( x_1, \ldots, x_{2n} \) of the integers \( 1, \ldots, 2n \) such that \( x_i + x_{i+1} \neq 2n + 1 \) for \( i = 1, \ldots, 2n - 1 \). (Problem 10G) (3 marks)

3) Prove that
\[ \sum_{n=1}^{\infty} S(n, n-2) x^n = \frac{x^3(1+2x)}{(1-x)^5}. \]  
(Problem 13K) (3 marks)

4) The Delannoy numbers \( d_{m,n} \) is the number of lattice paths in the Cartesian plane that start at \( (0,0) \), end at \( (m,n) \) using steps \( (0,1), (1,0) \) and \( (1,1) \) (e.g., see next question). It is easy to see (you need not prove this)
\[
\begin{align*}
    d_{m,n} & = \\
    & = \begin{cases} 
1 & \text{if } m+n \leq 1 \\
    d_{m,n-1} + d_{m-1,n} + d_{n-1,m-1} & \text{otherwise.}
\end{cases}
\end{align*}
\]
Use this recurrence relation to determine the generating function for the Delannoy numbers
\[ D(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{m,n} x^n y^m \]  
(2 marks)

5) The Schröder number \( S_n \) is the number of lattice paths in the Cartesian plane that start at \( (0,0) \), end at \( (n,n) \), contain no points above the line \( y = x \) using steps \( (0,1), (1,0) \) and \( (1,1) \). We demonstrate this by enumerating the \( S_3 = 22 \) cases below. One can easily see \( S_1 = 2 \) and \( S_2 = 6 \).

\[
\begin{align*}
\text{(a) Show that the Schröder numbers satisfy the recurrence relation} \\
S_n & = S_{n-1} + \sum_{k=0}^{n-1} S_k S_{n-1-k}. 
\end{align*}
\]
(b) Use the recurrence relation to show that the generating function for the Schröder number is
\[ S(x) = \sum_{n=0}^{\infty} S_n x^n = \frac{1 - \sqrt{1 - 6x + x^2}}{2x}. \]  
(3 marks)

Note : This is a more common form of Problem 14N.

6) The Hermite polynomials are specified by the three term recurrence relation
\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \]
with initial conditions \( H_0(x) = 1 \) and \( H_1(x) = 2x \). Determine the exponential generating function for these polynomials, i.e., determine a closed form for the function
\[ H(x, y) = \sum_{n=0}^{\infty} \frac{H_n(x)y^n}{n!}. \]
Use this to show \( H_n'(x) = 2nH_{n-1}(x) \). (3 marks)