Homework 3 : Ma121a - Combinatorics

Homework is due on Thursday the 6th of November at 12:00. While collaboration is encouraged, you must write your own solutions.

1) A perfect matching in a graph \( G \) (not necessarily bipartite) is a matching so that each vertex of \( G \) is incident with one edge of the matching. Let \( G \) be a simple graph in which every vertex has degree 3. Prove that \( G \) has a perfect matching if and only if there is a decomposition of \( G \) into 3-edge paths. (4 marks)

2) Let \( S \) be the set \( \{1, 2, \ldots, mn\} \). We partition \( S \) into \( m \) sets, \( A_1, A_2, \ldots, A_m \) of size \( n \). Let a second partitioning into \( m \) sets of size \( n \) be \( B_1, B_2, \ldots, B_m \). Show that the sets \( A_i \) can be renumbered in such a way that \( A_i \cap B_i \neq \emptyset \). (Problem 5D). (3 marks)

3) Let \( A_1, A_2, \ldots, A_n \) be finite sets. Show that if

\[
\sum_{1 \leq i < j \leq n} \frac{|A_i \cap A_j|}{|A_i| \cdot |A_j|} < 1,
\]

then the sets \( A_1, A_2, \ldots, A_n \) have a system of distinct representatives. (Problem 5F). (4 marks)

4) Let \( n^2 + 1 \) points be given in \( \mathbb{R}^2 \). Prove that there is a sequence of \( n + 1 \) points \((x_1, y_1), \ldots, (x_{n+1}, y_{n+1})\) for which \( x_1 \leq x_2 \leq \cdots \leq x_{n+1} \) and \( y_1 \geq y_2 \geq \cdots \geq y_{n+1} \) or a sequence of \( n + 1 \) points for which \( x_1 \leq x_2 \leq \cdots \leq x_{n+1} \) and \( y_1 \leq y_2 \leq \cdots \leq y_{n+1} \). (4 marks)

5) Construct the maximal flow for the transportation network below. (Problem 7A).

![Transportation Network Diagram]

The source is \( s \) and the sink is \( t \).

6) Show that the dimension of the vector space of all circulations on a connected digraph \( D \) is \( |E(D)| - |V(D)| + 1 \). (Problem 7F) (2 marks)

(3 marks)