Homework 1: Ma121a - Combinatorics

Homework is due on Thursday the 9th of October at 3:00pm. While collaboration is encouraged, you must write your own solutions.

1) Show that a finite simple graph with more than one vertex has at least two vertices with the same degree. (Problem 1G, Page 9) (1 mark)

2) Consider the following tree

   (a) Determine the Pfüffer sequence associated with the following tree. (2 marks)
   (b) Determine the function $f : \{2, \ldots, 10\} \rightarrow \{1, \ldots, 11\}$ associated with this tree. (2 marks)

3) (a) Draw the tree associated with the Prüfer code $(3\ 3\ 3\ 7\ 4\ 2\ 2\ 1)$ (2 marks)
   (b) Draw the tree associated with the function $f(2) = 5$ $f(3) = 7$ $f(4) = 5$ $f(5) = 2$
      $f(6) = 2$ $f(7) = 7$ $f(8) = 4$ $f(9) = 4$. (2 marks)

4) Consider the following weighted graph:

   Determine the cheapest spanning tree and its cost. (2 marks)

5) How many trees $T$ are there on the set of vertices $\{1, 2, 3, 4, 5, 6, 7\}$ in which the vertices 2 and 3 have degree 3, vertex 5 has degree 2, and hence all other vertices have degree 1? Do not just draw pictures but consider the possible Prüfer codes of these trees. (Problem 2B, Page 15) (2 marks)

6) Consider a rectangular chessboard of size $4 \times n$. Define a graph, $G$, on this chessboard where every position on the board is a node and any two nodes, $v_1$ and $v_2$, are connected by an edge if and only if a knight can move from $v_1$ to $v_2$. I.e, in any “L” shape of size $(2,1)$, as depicted below. Show that $G$ is not Hamiltonian. (4 marks)

The white knights indicate where the black knight may move.

7) If $G$ is a simple graph with $n(\geq 3)$ vertices, $v_1, \ldots, v_n$, with degrees $d_1, \ldots, d_n$, show that if $d_i + d_j \geq n$ for each pair of non-adjacent vertices, $v_i$ and $v_j$, then $G$ is Hamiltonian. (3 marks)