Ma/CS 6a

Class 5: Basic Counting

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

By Adam Sheffer

No Collaboration Problems

• Every assignment will have (at most) one problem marked **NO COLLABORATION**.
  ◦ These are problems that you are supposed to do on your own.
  ◦ No asking for hints in office hours either (asking for clarifications is OK).
  ◦ Usually medium difficulty problems.
### Permutations

- **Problem.** Given a set \( \{1, 2, \ldots, n\} \), in how many ways can we order it?

- **The case \( n = 3 \).** Six distinct orders / permutations: 123, 132, 213, 231, 312, 321.

- **The general case.**
  \[
  n! = n \cdot (n - 1) \cdot \cdots \cdot 2 \cdot 1
  \]

### Total Number of Subsets

- **Problem.** How many subsets does the set \( S = \{1, 2, \ldots, n\} \) have?
  - Two options for every element \( i \in S \). Either \( i \) is in the subset or not.
  - Since there are \( n \) element in \( S \), the number of subsets is \( 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n \).
Subsets of Size $k$

- Given a set $\{1,2, ..., n\}$, how many (unordered) **subsets of size $k$** does it have?
- Example. Consider the case $n = 5$ and $k = 3$.
  - The possible subsets are $(1,2,3), (1,2,4), (1,2,5), (1,3,4), (1,3,5), (1,4,5), (2,3,4), (2,3,5), (2,4,5), (3,4,5)$.
  - 10 distinct subsets!

Subsets of Size $k$ (cont.)

- Given a set $S = \{1,2, ..., n\}$, how many (unordered) subsets of size $k$ does it have?
- Look at the $n!$ orderings of $S$ and consider the first $k$ numbers as the subset.
  - For example, when $n = 5$ and $k = 3$
    - $123\,45\quad342\,51$
    - $135\,24\quad341\,52$
    - $543\,21\quad135\,42$
Binomial Coefficients

- Given a set $S = \{1, 2, \ldots, n\}$, how many (unordered) subsets of size $k$ does it have?
- Look at the $n!$ orderings of $S$ and consider the first $k$ numbers as the subset.
  - Every subset is obtained $k! \ (n-k)!$ times, so
  $$\binom{n}{k} = \frac{n!}{k! \ (n-k)!}$$

Pronounced “$n$ choose $k$”

Warm-up Problem

- **Prove or disprove.** For every $n \geq k \geq 0$
  $$\binom{n}{k} = \binom{n}{n-k}.$$ 
- **True.** Deciding which $k$ elements to choose is like deciding which $n-k$ elements not to take.
Pascal’s Rule

• **Prove.** For every \( n \geq k \geq 0 \)

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

\# of subsets containing 1 \hspace{2cm} \# of subsets not containing 1

Pascal’s Triangle

• Pascal’s rule: \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \).
• \( \binom{n}{k} \) is element \( k + 1 \) of row \( n + 1 \).

\[
\begin{array}{ccccccc}
& & & 1 & & & \\
& & 1 & & 1 & & \\
& 1 & & 2 & & 1 & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

*Every number is the sum of the two numbers above it.*
A Sum of Binomial Coefficients

• **Prove.** For every $n, k > 0$

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.
\]

• The left-hand side is the number of subsets of \{1,2,3,...,n\}, which is $2^n$.

Partitioning into $k$ Subsets

• **Problem.** For $n, k > 0$, we have $n$ identical balls and $k$ bins. In how many can place the balls in the bins?

• **Example.** If we have three balls and two bins, there are four options: (3,0), (2,1), (1,2), (0,3).
Partitioning into $k$ Subsets

- **Problem.** For $n \geq k \geq 0$, we have $n$ identical balls and $k$ bins. In how many can place the balls in the bins?

- **Answer.** $\binom{n + k - 1}{k - 1}$. The $k - 1$ choices correspond to the end of each bin.

```
  X   X   X   X   X
  •   •   •   •   •
```

<table>
<thead>
<tr>
<th>Bin #1:</th>
<th>Bin #2:</th>
<th>Bin #4:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ball</td>
<td>3 balls</td>
<td>empty</td>
</tr>
</tbody>
</table>

The Binomial Theorem

- **Recall.**
  - $(x + y)^2 = x^2 + 2xy + y^2$.
  - $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

- **The binomial theorem.** What is $(x + y)^n$?

$$\sum_{0 \leq i, j \leq n \atop i + j = n} \binom{n}{i} x^i y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots$$
The Binomial Theorem and Pascal’s Triangle

\[(x + y)^1 = x + y\]
\[(x + y)^2 = x^2 + 2xy + y^2\]
\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\]
\[(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]
\[
\begin{array}{cccc}
1 & & & \\
1 & 1 & & \\
1 & 2 & 1 & \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

The Binomial Theorem – Proof

- **The binomial theorem.**
  \[(x + y)^n = \sum_{0 \leq i \leq n} \binom{n}{i} x^i y^{n-i}.\]

- **Proof.** We have
  \[(x + y)^n = (x + y)(x + y)\cdots(x + y).\]
- The coefficient of \(x^i y^{n-i}\) is the number of ways to choose \(x\) from \(i\) of the parentheses and \(y\) from the remaining ones.
- That is, the coefficient of \(x^i y^{n-i}\) is \(\binom{n}{i}\).
Monomials and Degrees

- Polynomials are sums of monomials:
  \[ x^7 + 3x^2y^4z + 5x^3z^3 + \cdots \]

- The degree of a monomial is the sum of the powers of its variables.
  \[ \text{deg}(3x^2y^4z) = 2 + 4 + 1 = 7. \]

- The degree of a polynomial is the maximum of the degrees of its monomials
  \[ \text{deg}(x^5 + 3x^2y^4z + 5x^3z^3) = 7 \]

Number of Monomials

- **Problem.** How many distinct monomials can a polynomial of degree \( D \) in \( k \) variables have?

- **Answer.** Take \( k + 1 \) bins – one for every variable and one extra. Every placement of \( D \) balls in the bins corresponds to a monomial.
Number of Monomials

- **Problem.** How many distinct monomials can a polynomial of degree $D$ in $k$ variables have?

- **Answer.** Take $k + 1$ bins – one for every variable and one extra. Every placement of $D$ balls in the bins corresponds to a monomial.

\[
\binom{D + k}{k}
\]

Returning to Lecture 3

- To prove “Fermat’s little theorem”, we assumed, without proof, that for any prime $p$

\[(a + b)^p \equiv a^p + b^p \mod p.\]

- **Proof.** By the binomial theorem:

\[(a + b)^p = \binom{p}{0}a^p + \binom{p}{1}a^{p-1}b + \binom{p}{2}a^{p-2}b^2 + \cdots\]

- To prove the claim, it suffices to prove that $p\mid \binom{p}{i}$ for every $1 \leq i \leq p - 1$.

- This holds since in $\binom{p}{i} = \frac{p!}{i!(p-i)!}$ the numerator is divisible by $p$ but the denominator is not.
Partitions of an Integer

- \( r, n \) – two positive integers.
- **Problem.** What is the number of solutions of
  \[ a_1 + a_2 + \cdots + a_r = n, \]
  where each \( a_i \) is a natural number?

\[
5 = 1 + 1 + 3 = 1 + 3 + 1 = 0 + 0 + 5 = 1 + 0 + 4 = \ldots
\]

Solution

- Consider \( n \) as a sum of \( n \) unit elements.
- Dividing these elements across the \( r \) variables \( a_i \) is equivalent to placing \( n \) balls in \( r \) bins.
  - The value of \( a_i \) is the number of balls in the \( i \)'th bin.

\[
\binom{n + r - 1}{r - 1}
\]
Another Inequality

**Problem.** Prove the identity
\[
\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.
\]

**Proof.**

- We begin with the identity
  \[(1 + x)^n(1 + x)^n = (1 + x)^{2n}.
  \]
- By the binomial theorem, we have
  \[
  \left(\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n\right)\left(\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n\right)
  = \left(1 + \binom{2n}{1} x + \cdots + \binom{2n}{2n} x^{2n}\right).
  \]

**Proof (cont.)**

\[
\left(\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n\right)\left(\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n\right)
 = \left(1 + \binom{2n}{1} x + \cdots + \binom{2n}{2n} x^{2n}\right).
\]

- Consider the coefficient of \(x^n\) on each side.

  - On the right hand side, it is \(\binom{2n}{n}\).
  - On the left hand side, it is
    \[
    \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}
    = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.
    \]
Summing Up

• In how many ways can we choose \( k \) elements from \( \{1, 2, 3, \ldots, n\} \)?

<table>
<thead>
<tr>
<th></th>
<th>Ordered</th>
<th>Unordered</th>
</tr>
</thead>
<tbody>
<tr>
<td>No repetitions</td>
<td>( \frac{n!}{(n-k)!} )</td>
<td>( \binom{n}{k} )</td>
</tr>
<tr>
<td>With repetitions</td>
<td>( n^k )</td>
<td>( \binom{k+n-1}{n-1} )</td>
</tr>
</tbody>
</table>

Summing Up #2

• In how many ways can we place \( k \) balls into \( n \) bins?

<table>
<thead>
<tr>
<th></th>
<th>At most 1 ball in each bin</th>
<th>Any number of balls in each bin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Each ball has a different color</td>
<td>( \frac{n!}{(n-k)!} )</td>
<td>( n^k )</td>
</tr>
<tr>
<td>Balls are indistinguishable</td>
<td>( \binom{n}{k} )</td>
<td>( \binom{k+n-1}{n-1} )</td>
</tr>
</tbody>
</table>
The End

Imagine that you’re drawing at random from an urn containing fifteen balls—six red and nine black.

OK, I reach in and... ...my grandfather’s ashes??! Oh God!

I... what?

Why would you do this to me??!