Recall: Power Series

- We define sums and products of power series as in the case of polynomials.
  - $A(x) = a_0 + a_1x + a_2x^2 + \cdots$
  - $B(x) = b_0 + b_1x + b_2x^2 + \cdots$
  - $A(x) + B(x)$
    \[ = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots \]
  - $A(x)B(x)$
    \[ = (a_0b_0) + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots \]
Reminder: Inverse Elements

- **Theorem.** A power series
  \[
  A(x) = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{R}[[x]]
  \]
  has an inverse if and only if \(a_0 \neq 0\).

- **Example.**
  - What is the inverse of \(1 - x\)?
    \[
    (1 - x)A(x) = 1.
    
    A(x) = 1 + x + x^2 + x^3 + \cdots
    \]

Negative Exponents Formula

- **Theorem.** For any positive integer \(m\),
  \[
  (x + 1)^{-m} = \sum_{n \geq 0} (-1)^n \binom{m + n - 1}{n} x^n
  \]

- **Examples.**
  - \((x + 1)^{-1} = \sum_{n \geq 0} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots\)
  - \((x + 1)^{-2} = \sum_{n \geq 0} (-1)^n (n + 1) x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots\)
Generating Functions

- Given an infinite sequence of numbers $a_0, a_1, a_2, \ldots$, the \textit{generating function} of the sequence is the power series $a_0 + a_1 x + a_2 x^2 + \cdots$

- \textbf{Example.}
  - Recall the \textit{Fibonacci numbers}:
    $$F_0 = F_1 = 1 \quad F_i = F_{i-1} + F_{i-2}.$$  
  - The corresponding generating function is
    $$1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

Why Mathematicians Should \textbf{Not} Watch Disney’s 1939 Snow White and the Seven Dwarfs

\[\star\]
Helping Adam Make Money

- **Problem.** The story of the millionaire Adam:
  - He started with nothing, and after working hard for a year managed to get 1$. 
  - After the second year he had 5$. 
  - Afterwards, at the beginning of each year Adam bought assets of value six times his worth at the beginning of the previous year. 
  - At the end of each year Adam sold these assets for four times his worth at the beginning of the year. 
  - **How many years did it take Adam to become a millionaire?**

Rephrasing with Generating Functions

- Consider the generating function of the problem $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ ($a_i$ is the money at the end of $i$'th year).
- We already know $a_0 = 0, a_1 = 1, a_2 = 5$.
- What other information do we have? $a_i = a_{i-1} - 6a_{i-2} + 4a_{i-3}$, for $i \geq 2$.
- Rearranging and replacing $i$ with $i + 2$: $a_{i+2} - 5a_{i+1} + 6a_i = 0$. 
Using the Recurrence Relation

By applying the *recurrence relation*
\[ a_{i+2} - 5a_{i+1} + 6a_i = 0, \]
we have
\[
A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots
\]
\[ = 0 + x + x^2(5a_1 - 6a_0) \\
+ x^3(5a_2 - 6a_1) + \cdots \\
= x + 5(a_1x^2 + a_2x^3 + \cdots) \\
- 6(a_0x^2 + a_1x^3 + \cdots) \\
= x + 5xA(x) - 6x^2A(x).
\]

Looking for \( A(x) \)

We have
\[ A(x) = x + 5xA(x) - 6x^2A(x). \]
That is,
\[
A(x) = \frac{x}{1 - 5x + 6x^2} = \frac{1}{1 - 3x} - \frac{1}{1 - 2x}.
\]

Last time we proved
\[
(1 - ax)^{-m} = \sum_{n \geq 0} \binom{m + n - 1}{n} a^n x^n.
\]
\[
\circ (1 - 2x)^{-1} = \sum_{n \geq 0} 2^n x^n.
\]
\[
\circ (1 - 3x)^{-1} = \sum_{n \geq 0} 3^n x^n.
\]
Solving the Problem

\[ A(x) = \frac{1}{1 - 3x} - \frac{1}{1 - 2x} \]
\[ = \sum_{n \geq 0} 3^n x^n - \sum_{n \geq 0} 2^n x^n \]
\[ = \sum_{n \geq 0} (3^n - 2^n)x^n. \]

- That is, after the \( i \)\(^{\text{th}} \) year, Adam had \( 3^n - 2^n \). Millionaire after the 13\(^{\text{th}} \) year.

Generating Functions and Algorithms

- We wish to compute some property of a graph.
  - If the graph has one vertex, we can compute this property in \( 1 \mu s \).
  - If the graph has \( n > 1 \) vertices, we cut the graph into two parts of \( \sim n/2 \) points that can be handled separately. This takes \( 50n \mu s \).

- How long does it take to handle a graph with \( n \) vertices?
  - Solve \( a_m = 50n + 2a_{n/2} \) and \( a_1 = 1 \).
Alan Turing

- English mathematician. Known for:
  - Invented the **Turing machine** (and thus helped formalizing the idea of algorithms).
  - Discovered the **halting problem**.
  - One of the main breakers of the **Enigma code** in World War II.
  - Invented the **Turing test** (and thus sometimes considered as the father of artificial intelligence).

Alan Turing’s Death

- Turing was obsessed with Disney’s **Snow white and the seven dwarfs**.
- He especially liked citing the wicked witch’s lines about giving snow white the poisoned apple.
- On June 8th 1954, Turing committed suicide by biting a poisoned apple (he injected cyanide into it).
Recap: Using Generating Functions

- Solving problems via generating functions:
  - In the problem, we identify the first few values of $a_0, a_1, a_2, \ldots$, and also a recursion relation.
  - We use these to obtain an equation of the form $A(x) = x + 5xA(x) - 6x^2A(x)$.
  - Isolate $A(x)$ to obtain an expression of the form $A(x) = \frac{x}{1-5x+6x^2}$.
  - Simplify the expression to obtain a sum of “simple” terms, each which can be written as a power series.

A Problematic Step

- In the millionaire question, we had the step $\frac{x}{1-5x+6x^2} = \frac{1}{1-3x} - \frac{1}{1-2x}$.
- Can we rewrite every fraction as a sum of “nice” parts?
  - Not when working over $\mathbb{R}$:
    $\frac{1}{x^2 + 2x + 8}$ = ???
  - If we work over $\mathbb{C}$, every polynomial is a product of linear terms:
    $x^2 + 2x + 8 = (x + 1 + i\sqrt{7})(x + 1 - i\sqrt{7})$. 
Polynomial Division

- **Recall.** Given two integers $a, b \in \mathbb{Z}$, there are unique $q, r \in \mathbb{Z}$ such that $r < b$ and
  $$a = qb + r.$$  

- **The polynomial variant.** Given two polynomials $a(x)$ and $b(x) \neq 0$, there are unique $q(x), r(x)$ such that
  $$a(x) = q(x)b(x) + r(x).$$  
  and either $\deg r(x) < \deg b(x)$ or $r(x) = 0$.

Example: Polynomial Division

- Divide $a(x) = x^4 + 3x^3 - 2x^2 + 5$ by $b(x) = x^2 - 2x + 1$.
  $$a(x) = q(x)b(x) + r(x)$$  
  $$a(x) = x^2 \cdot b(x) + 5x^3 - 3x^2 + 5$$  
  $$a(x) = (x^2 + 5x) \cdot b(x) + 7x^2 - 5x + 5$$  
  $$a(x) = (x^2 + 5x + 7) \cdot b(x) + 9x - 2$$
  
  $q(x) = x^2 + 5x + 7$ \text{ and } $r(x) = 9x - 2$
Larger Numerator

- We wish to write expressions of the form \( \frac{a(x)}{b(x)} \) as sums of “simple” terms.
- If \( \deg a(x) \geq \deg b(x) \), by the division property we can rewrite the expression as \( q(x) + \frac{r(x)}{b(x)} \).
- Thus, it suffices to consider cases where \( \deg a(x) < \deg b(x) \).

GCD of Polynomials

- A polynomial \( a(x) \) is a divisor of a polynomial \( b(x) \) if there exists a polynomial \( f(x) \) such that \( b(x) = a(x)f(x) \).
- The greatest common divisor of two polynomials \( a(x), b(x) \) (denoted \( \text{GCD}(a, b) \)) satisfies:
  - \( \text{GCD}(a, b) \) is a divisor of both \( a(x) \) and \( b(x) \).
  - Any other divisor of both \( a(x) \) and \( b(x) \) is also a divisor of \( \text{GCD}(a, b) \).
GCD Property

- **Recall.** For any $a, b \in \mathbb{Z}$, there exist $s, t \in \mathbb{Z}$ such that
  \[ \text{GCD}(a, b) = as + bt. \]

- **Theorem.** For any two polynomials $a(x), b(x)$, there exist polynomials $s(x), t(x)$ such that
  \[ \text{GCD}(a, b) = a(x)s(x) + b(x)t(x). \]

  \[
  \text{GCD}(x^4(x - 1)^2(x + 1), x^5(x + 1)^3(x + 4)^2) = x^4(x + 1).
  \]

Partial Fractions

- **Theorem.** Consider polynomials $a(x), b(x)$ so that
  - $\deg a(x) < \deg b(x)$.
  - $b(x)$ has a nonzero constant term.
  - $b(x) = s(x)t(x)$, such that $\text{GCD}(s, t) = 1$.

  Then there exist $f(x), g(x)$ such that
  \[
  \frac{a(x)}{b(x)} = \frac{f(x)}{s(x)} + \frac{g(x)}{t(x)},
  \]
  \[ \deg f(x) < \deg s(x) \text{ and } \deg g(x) < \deg t(x). \]
Example: Partial Fractions

- Consider the expression
  \[ \frac{a(x)}{b(x)} = \frac{5 - 3x}{x^2 - 3x + 2}. \]
  - \(b(x)\) has a nonzero constant term.
  - \(\text{deg } b(x) > \text{deg } a(x)\).
  - \(b(x) = (x - 1)(x - 2)\).

\[
\frac{a(x)}{b(x)} = \frac{5 - 3x}{x^2 - 3x + 2} = -\frac{2}{x - 1} - \frac{1}{x - 2}.
\]

Proof

- For simplicity, we write \(b = st\) (etc.).
  - Since \(GCD(s, t) = 1\), there exist polynomials \(u, v\) such that
    \[ 1 = su + tv \Rightarrow a = asu + atv. \]
  - Dividing \(av\) by \(s\), we have
    \[ av = qs + r, \]
    where the remainder satisfies \(\text{deg } r < \text{deg } s\).

\[
\frac{a}{b} = \frac{asu + atv}{st} = \frac{asu + t(qs + r)}{st}.
\]
Proof (cont.)

- We have \( \frac{a}{b} = \frac{asu + t(qs + r)}{st} \), where \( \deg r < \deg s \).

- Setting \( f = r \) and \( g = au + tq \), we have
  \[
  \frac{a}{b} = \frac{asu + t(qs + r)}{st} = \frac{sg + tf}{st} = \frac{f}{s} + \frac{g}{t}.
  \]

- It remains to bound \( \deg f \) and \( \deg g \):
  - \( \deg f = \deg r < \deg s \).
  - \( \deg ft < \deg st = \deg b \).
  - Since \( a = tf + sg \), we have \( \deg sg = \deg(a - ft) < \deg b = \deg st \).

Using Partial Fractions

- We wish to decompose \( \frac{a(x)}{b(x)} \) where
  \[ \deg a(x) < \deg b(x) \text{ and } b(x) = p_1(x)^{m_1}p_2(x)^{m_2} \cdots p_d(x)^{m_d}. \]

- By repeatedly applying the partial fractions technique, we obtain
  \[
  \frac{a(x)}{b(x)} = \frac{h_1(x)}{p_1(x)^{m_1}} + \frac{h_2(x)}{p_2(x)^{m_2}} + \cdots + \frac{h_d(x)}{p_d(x)^{m_d}}
  \]
  (where \( \deg h_i(x) < m_i \deg p_i(x) \)).
Recap: Simplifying $a(x)/b(x)$

- We decompose $a(x)/b(x)$ to
  \[
  \frac{a(x)}{b(x)} = \frac{h_1(x)}{p_1(x)^{m_1}} + \frac{h_2(x)}{p_2(x)^{m_2}} + \ldots + \frac{h_d(x)}{p_d(x)^{m_d}}.
  \]

- Assume that $p_i(x)$ is linear. That is, $p_i(x) = (ax + b)$. Then we have
  \[
  \frac{h_i(x)}{p_i(x)^{m_i}} = \frac{h_i(x)}{a^{m_i}} (x + b/a)^{-m_i},
  \]
  and we already know how to compute $(x + b/a)^{-m_i}$.

Kurt Gödel

- An Austrian mathematician.
  - Considered as one of the top logicians ever.
  - Famous for his two incompleteness theorems (which also pushed Turing to come up with the Turing machine and the halting problem).
The Death of Kurt Gödel

• Gödel was also obsessed with Disney’s snow white and the seven dwarfs. He used to try to convince his good friend Albert Einstein to see the movie with him.
  ◦ Due to the movie, Gödel became paranoid about people trying to poison his food. He only agreed to eat his wife’s cooking.
  ◦ When his wife was hospitalized for several months, he died of starvation.

Who will be Next?

Stop Disney before it is too late!
Finding the Power Series

- **Problem.** Find the **power series** of
  \[ G(x) = \frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1}. \]

- **Solution.**
  - First we **factorize**
    \[ 9x^3 - 9x^2 - x + 1 = (1-x)(1-3x)(1+3x). \]
  - Thus, we would like to rewrite the expression:
    \[ \frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1} = \frac{A}{1-x} + \frac{B}{1-3x} + \frac{C}{1+3x}. \]

Finding the Partial Fractions

- We would like to solve
  \[ \frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1} = \frac{A}{1-x} + \frac{B}{1-3x} + \frac{C}{1+3x}. \]

- Multiply both sides by \((1-x)(1-3x)(1+3x)\):
  \[ 12x^2 - 24x + 4 = A(1-3x)(1+3x) + B(1-x)(1+3x) + C(1-x)(1-3x). \]

- **Equating coefficients of \(x^2, x, 1\), we obtain**
  \[ 12 = -9A - 3B + 3C, \]
  \[ -24 = 2B - 4C, \]
  \[ 4 = A + B + C. \]
Finding the Partial Fractions (cont.)

- We would like to solve
  \[
  \frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1} = \frac{A}{1-x} + \frac{B}{1-3x} + \frac{C}{1+3x}.
  \]

- We have
  \[
  12 = -9A - 3B + 3C, \\
  -24 = 2B - 4C, \\
  4 = A + B + C.
  \]

- The solution to the system is \( A = 1, B = -2, C = 5 \), so
  \[
  G(x) = \frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1} = \frac{1}{1-x} - \frac{2}{1-3x} + \frac{5}{1+3x}.
  \]

The Power Series

- We have \( G(x) = (1 - x)^{-1} - 2(1 - 3x)^{-1} + 5(1 + 3x) \). Recall that
  \[
  (1 - ax)^{-1} = \sum_{n \geq 0} a^n x^n.
  \]

  \[
  (1 - x)^{-1} = \sum_{n \geq 0} x^n, \quad (1 - 3x)^{-1} = \sum_{n \geq 0} 3^n x^n, \\
  (1 + 3x)^{-1} = \sum_{n \geq 0} (-3)^n x^n.
  \]

- We thus have
  \[
  G(x) = \sum_{n \geq 0} (1 - 2 \cdot 3^n + 5(-3)^n)x^n.
  \]
Next Week...