Ma 5a: Midterm Practice Problems

1. Assume $H$ and $K$ are subgroups of $G$.
   
   (a) Show that if $K \subseteq H$ then $N_H(K) = N_G(K) \cap H$.
   
   (b) Show that if $K \trianglelefteq G$ then $H \cap K \trianglelefteq H$.

2. Show that if $H$ and $K$ are normal subgroups of $G$. Then $HK$ is a normal subgroup of $G$. Show that the same is true if $H$ is only a subgroup of $G$ but not normal.

3. Let $\varphi : G \to G'$ be a homomorphism. Show that if $H'$ is a subgroup of $G'$, then $\varphi^{-1}(H)$ is a subgroup of $G$ containing $\ker \varphi$.

4. Assume $G$ is a finite abelian group. Show that the map $a \mapsto a^n$ is an automorphism of $G$ if and only if $(m, |G|) = 1$. (Hint: Use Lagrange Theorem to avoid tedious argument.)

5. Show that if $H$ is a normal subgroup of $S_4$ which contains $(13)$ and $(1234)$ then $H = S_4$.

6. Show that if $a$ and $b$ are two elements in an abelian group with order 3 and 5 respectively, then $ab$ has order 15.

7. Show every element in $S_5$ has order a divisor of 120.

8. Draw the subgroup lattice for $\mathbb{Z}/24\mathbb{Z}$ with indices marked. Is this group isomorphic to $S_4$?

9. Find the index of $< (12), (1234) >$ in $S_5$.

10. If $H$ is a subgroup of $G$ and $[G : H]$ is a prime. Show that any subgroups of $G$ containing $H$ is either equal to $H$ or $G$.

11. Assume $G$ is a group acting on the set $X$. Consider $\mathbb{C}(X)$, the set of all complex-valued functions $f : X \to \mathbb{C}$ on $X$. For any $g \in G$, $f \in \mathbb{C}(X)$, define $g.f : X \to \mathbb{C}$ by $(g.f)(x) := f(g^{-1}x)$. Show that this defines an action of $G$ on $\mathbb{C}(X)$.

12. Let $G$ be a group that acts on a nonempty set $X$. For $x \in X$, define a subset of $G.x = \{ g.x \in X \mid g \in G \}$ called the orbit of $x$ under the action of $G$. Let $\sim$ be a relation on $X$ such that $x \sim y$ if and only if $y \in G.x$. Show that $\sim$ is an equivalent relation on $X$ with equivalence classes the set of orbits $G.x$, $x \in X$.

13. Find the number of elements in $S_7$ which are conjugate to $(23)(47)$.

14. Describe the set of left cosets of $V_4 = \{ 1, (12)(34), (13)(24), (14)(23) \}$ in $S_4$. Is $V_4$ normal in $S_4$?

15. Show that the group $\mathbb{Z}/30\mathbb{Z}$ can be generated by $\bar{6}, \bar{10}, \bar{15}$ elements.

16. Show that if $|G| = 289$, then the group $G$ is abelian. (Hint: Show that $Z(G) = G$.)