1. Consider the differential equation
\[ \frac{dy}{dt} = r y \log \left( \frac{K}{y} \right), \]
with two positive constants \( r > 0 \) and \( K > 0 \).
- Find the equilibrium solutions
- Draw the graph of the function \( f(y) = r y \log(K/y) \)
- Describe qualitatively the behavior of the other solutions
- Are the equilibrium solutions stable or unstable?

2. The Riccati equation is a nonlinear first order differential equation of the form
\[ \frac{dx}{dt} = a(t) + b(t)x + c(t)x^2, \]
where \( a(t), b(t) \) and \( c(t) \) are assigned functions.
- Show that, if \( \phi(t) \) is a solution to the equation, and \( u(t) \) is a solution to the linear first order equation
\[ \frac{du}{dt} = -(b(t) + 2c(t)\phi(t))u - c(t), \]
then the function
\[ x(t) = \phi(t) + \frac{1}{u(t)} \]
is also a solution of the Riccati equation.
• Use this to solve the Riccati equation

\[ x' = 1 + t^2 - 2tx + x^2. \]

(Hint: notice that \( \phi(t) = t \) is a solution.)

3. Solve the previous equation in a simpler way by viewing it as:

\[ \frac{dx}{dt} = 1 + (x - t)^2. \]

4. Given an equation of the form \( P(t)y'' + Q(t)y' + R(t)y = 0 \), let \( \mu(t) \) be an integrating factor such that the equation

\[ M(t)y'' + N(t)y' + S(t)y = 0, \tag{1} \]

with \( M(t) = \mu(t)P(t), \ N(t) = \mu(t)Q(t) \) and \( S(t) = \mu(t)R(t) \), can be written as

\[ (M(t)y')' + (f(t)y)' = 0, \tag{2} \]

for a function \( f(t) \) that depends on \( M(t), N(t), \) and \( S(t) \).

- Write the solutions of (1) in terms of \( M(t), f(t) \) and integrals.
- Show that the integrating factor \( \mu(t) \) should satisfy the adjoint equation

\[ P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0. \]

- Show that the adjoint equation of the adjoint equation is the original equation \( Py'' + Qy' + Ry = 0. \)
- Compute the adjoint equation of the Airy equation \( y'' - ty = 0 \) and of the Bessel equation \( t^2y'' + ty' + (t^2 - \nu^2)y = 0. \)