Midterm Exam 2015 solutions: Do not read if you have not taken midterm exam.

1a. We will first show by induction that any polynomial \( p \) of degree \( n \geq 1 \) is equal to its Taylor Series about \( c \). (This was essentially done in Cranks section 1.2, although the induction was suppressed.) First we do the base case. Let \( p(x) = ax + b \) be of degree 1. Then \( p(c) = ac + b \). And we have the formula,

\[
p(x) = a(x - c) + ac + b = p'(c)(x - c) + p(c),
\]

which is the Taylor expansion.

Next we suppose the induction hypothesis, namely that if \( p(x) \) is of degree \( n \), then

\[
p(x) = p(c) + \sum_{j=1}^{n} \frac{1}{j!} p^{(j)}(c)(x - c)^j.
\]

Now suppose \( p(x) = ax^{n+1} + p_{\text{lower}}(x) \), where \( p_{\text{lower}} \) is a polynomial of degree at most \( n \). Observe that \( p^{(n+1)}(c) = (n + 1)!a \). Let

\[
a(x) = p(x) - a(x + c)^{n+1}.
\]

Note that \( a(x) \) is of degree at most \( n \) and that for any natural number \( j \leq n \), it is the case that

\[
a^{(j)}(c) = p^{(j)}(c),
\]

since the \( j \)th derivative of \( (x - c)^{n+1} \) vanishes at \( c \). Note also that \( a(c) = p(c) \). Now we apply the induction hypothesis to \( a(x) \), concluding that

\[
p(x) = p(c) + \sum_{j=1}^{n+1} \frac{1}{j!} p^{(j)}(c)(x - c)^j.
\]

This actually solves the problem by letting \( r = p(c) \) and

\[
q(x) = \sum_{j=1}^{n+1} \frac{1}{j!} p^{(j)}(c)(x - c)^{j-1}.
\]

1b. In this problem, we are trying to explain how polynomial division works.

We begin with the base case. Let \( p(x) \) be a polynomial of degree 1 and \( s(x) \) a nonzero polynomial of degree either 0 or 1. If \( s(x) \) is of degree 0, then it’s just a number,
s(x) = s. Then r(x) = 0 and q(x) = \frac{p(x)}{s}. If s(x) is of degree 1, then it can be written as s(x) = s'(0)(x - b), for some value b and we can reduce the problem to the previous part.

Now we assume the induction hypothesis. If p(x) is of degree at most n and s(x) is nonzero and of degree m ≤ n, then there is q(x) of degree at most n − m and r(x) of degree at most m − 1 with p(x) = s(x)q(x) + r(x).

Now let p(x) be of degree n + 1. Then p(x) = ax^{n+1} + p_{lower}(x), where p_{lower}(x) is a polynomial of degree at most n. Further s(x) = bx^m + s_{lower}(x), with m ≤ n + 1. The first term of q should be \frac{a}{b}x^{n+1−m}. Thus we define

\[ p_{new}(x) = p(x) − \frac{a}{b}x^{n+1−m}s(x). \]

Now there are two cases. In the first case, p_{new} is of lower degree than m. In this case, we can’t apply the induction hypothesis. But there’s no problem. We let q(x) = \frac{a}{b}x^{n+1−m}, and we let r(x) = p_{new}(x).

In the other case, p_{new}(x) has degree at most n but larger than the degree of s, and we use the induction hypothesis to write

\[ p_{new}(x) = s(x)q_{new}(x) + r(x), \]

and we let

\[ q(x) = \frac{a}{b}x^{n+1−m} + q_{new}(x). \]

2 a) We want to show that \( f(x) = \sqrt{x} \) is continuous at \( x = 1 \). For \( x > 1 \), we use the inequality

\[ 1 ≤ \sqrt{x} ≤ 1 + \frac{x − 1}{2}. \]

We see this just by squaring both sides. For \( 0 < x < 1 \), we use the inequality

\[ 1 ≥ \sqrt{x} ≥ x. \]

Now for any \( \epsilon > 0 \), we need to pick a \( \delta \) so that \( |x - 1| < \delta \) implies that \( |\sqrt{x} - 1| < \epsilon \). We just pick \( \delta \) to be the minimum of \( \epsilon \) and 1, and apply the inequalities.

2b) Observe that \( \sqrt{3 + \frac{1}{n}} = \sqrt{3} + \sqrt{1 + \frac{1}{3n}} \).

Now note that the sequence \( 1 + \frac{1}{3n} \) converges to 1. Since the function \( \sqrt{x} \) is continuous at 1, we then have that \( \sqrt{1 + \frac{1}{3n}} \) converges to \( \sqrt{1} \) which is 1. Therefore \( \sqrt{3 + \frac{1}{n}} \) converges to \( \sqrt{3} \). Now we use the fact that any convergent sequence is a Cauchy sequence.
3 a) Observe first that \( f(x) = x^2 - 20x + 101 \). Thus for \( x \geq 6 \), we have \( f(x) < x^2 \).

\[
\sum_{n=6}^{\infty} \frac{1}{f(n)} > \sum_{n=6}^{\infty} \frac{1}{n},
\]

so that the series does not converge absolutely.

Next observe that for \( x > 10 \), the function \( f(x) \) is increasing, since it has positive derivative. Then

\[
\sum_{n=11}^{\infty} \frac{-1^n}{f(n)},
\]

is convergent by the theorem on decreasing alternating series. Thus

\[
\sum_{n=1}^{\infty} \frac{-1^n}{f(n)},
\]

is convergent since convergence is determined by the tail. Since it is not absolutely convergent, it is conditionally convergent.

3 b) For any number \( k \), there are \( 910^{k-1} \) numbers \( n \) with \( l(n) = k \) and each is smaller than \( 10^k \). Thus each of these groups contributes at least \( \frac{9}{10} \) to the sum. Since there are infinitely many, the sum diverges.

4 a) Observe that since \( f \) is continuous it achieves a maximum \( M \) with \( f(x_M) = M \) and a minimum \( m \) with \( f(x_m) = m \) on the interval \([0,1]\). (That is \( x_m \in [0,1] \) and \( x_M \in [0,1] \).) Note that

\[
m \leq A \leq M.
\]

If equality holds in either case, we are done since we can take the point where the maximum or minimum is achieved. Otherwise we apply the Intermediate value theorem to find a point between \( x_m \) and \( x_M \) where \( f \) takes the value \( A \).

b) Suppose there is a solution \( 1 < c < 10 \) to

\[
c^4 - 1 = 5000c - 1.
\]

Let \( f(x) = x^4 \). Apply the Mean Value Theorem to \( f \) on the interval \([1,c]\). There is a point \( d \) on \((1,c)\), where

\[
f'(d) = 5000.
\]

However,

\[
f'(x) = 4x^3.
\]

Thus

\[
f(d) \leq 4000,
\]

3
for any $d \in [1, 10]$. We have reached a contradiction.

5 a) Use the continuity of $f'$ to choose $\epsilon > 0$ so that $f'(x) > \frac{1}{2}$ when $|x| \leq \epsilon$. Then by the mean value theorem, $f$ is increasing on $[-\epsilon, \epsilon]$.

b) First solution Let $f_j(x)$ be defined on $2^{-j} \leq x \leq 2^{1-j}$ as follows.

\[ f_j(x) = x + b_j(x), \]

where $b_j(x) = -2(x - 2^{-j})$ for $2^{-j} \leq x \leq 2^{-j} + 2^{-2j}$ and $b_j(x) = 2(x - 2^{-j} - 2^{-2j}) - 2^{-2j}$ for $2^{-j} + 2^{-2j} < x \leq 2^{-j} + 2^{1-2j}$ and 0 elsewhere.

Now define $g(x) = f_j(x)$ when $2^{-j} \leq x \leq 2^{1-j}$ and $g(x) = -f_j(-x)$ when $-2^{1-j} \leq x \leq -2^{-j}$ whenever $j > 2$ and $g(x) = x$ elsewhere. Away from $x = 0$, we have that $g(x)$ is continuous since the functions $b_j$ always vanish at the endpoints of the intervals.

We have the estimate $|b_j(x)| = O(2^{-2j})$ as $j \to \infty$. Thus

\[ g(x) = x + O(x^2) = x + o(x). \]

Therefore $g'(0) = 1$. This implies that $g$ is continuous at 0. Now take any $\epsilon > 0$. The interval $[-\epsilon, \epsilon]$ contains some interval $[2^{-j}, 2^{1-j}]$ on which $g$ is not increasing.

Second solution

Let

\[ g(x) = x + 10x^2 \sin\left(\frac{1}{x}\right). \]

Observe that $g(x) = x + o(x)$ and therefore $g'(0) = 1$. Elsewhere

\[ g'(x) = 1 - 10 \cos\left(\frac{1}{x}\right). \]

Thus in any interval $[-\epsilon, \epsilon]$ there are points where the derivative of $g$ is negative. Because the derivative is continuous at these points, we can find a subinterval where $g$ is decreasing.