HOMEWORK 4 SOLUTIONS

Problem 1

We want to show that $fg$ is $o(h^2)$ as $h \to 0$ where $f$ is $o(h)$ and $g$ is $O(h)$ as $h \to 0$. It suffices to show that for any $\varepsilon > 0$ there is a $\delta > 0$ so that $|h| < \delta \Rightarrow \left| \frac{f(h)g(h)}{h^2} \right| < \varepsilon$. Pick $\delta_1 > 0$ so that $|h| < \delta_1 \Rightarrow |g(h)| < C|h|$ and pick $\delta_2$ so that $|h| < \delta_2 \Rightarrow |f(h)| < \varepsilon|h|/C$. Now set $\delta = \min\{\delta_1, \delta_2\}$ Then for $|h| < \delta$ we have:

$$\left| \frac{f(h)g(h)}{h^2} \right| = \left| \frac{f(h)}{|h|^2} \right| |g(h)| \leq \frac{\varepsilon C|h|^2}{C|h|^2} = \varepsilon$$

so we are done.

Problem 2

Taylor’s approximation (second order),

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2} f''(c) + o(h^2)$$

Since, $h$ is arbitrary, we also have,

$$f(c-h) = f(c) - hf'(c) + \frac{h^2}{2} f''(c) + o(h^2)$$

$$\Rightarrow f(c+h) + f(c-h) = 2f(c) + h^2 f''(c) + o(h^2)$$

[using the fact $o(h^2) + o(h^2) = o(h^2)$]

$$\Rightarrow f(c + h) + f(c - h) - 2f(c) - h^2 f''(c) = o(h^2)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(c + h) + f(c - h) - 2f(c) - h^2 f''(c)}{h^2} = 0 \quad \text{[by definition of $o(h^2)$]}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(c + h) + f(c - h) - 2f(c) - h^2 f''(c)}{h^2} - f''(c) = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(c + h) + f(c - h) - 2f(c)}{h^2} = f''(c)$$

Problem 3

Let $f(x)$ be a function on an interval $(a, b)$ which is differentiable at $c \in (a, b)$. Show in full detail (there’s a hint in the notes) that $f$ is continuous at $c$.

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1. Version 1: Using the First Definition of the Limit

By the definition of the derivative [lecture 8 notes, page 1] we know, that
\[
\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = f'(c).
\]
Therefore we have
\[
\lim_{x \to c} (f(x) - f(c)) = \lim_{h \to 0} (f(c + h) - f(c)) = \left( \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \right) \left( \lim_{h \to 0} h \right) = f'(c) \cdot 0 = 0.
\]
Thus
\[
\lim_{x \to c} f(x) = f(c) + \lim_{x \to c} (f(x) - f(c)) = f(c),
\]
and, therefore, \( f \) is continuous at \( c \).

2. Version 2: Using the Differential Approximation

Using differential approximation [lecture 8 notes, page 1] we know, that
\[
\lim_{x \to c} f(x) = \lim_{h \to 0} f(c + h) = \lim_{h \to 0} (f(c) + h f'(c) + o(h)) = f(c) + f'(c) \lim_{h \to 0} h + \lim_{h \to 0} o(h).
\]
Note, that this computation is correct if the limits in the right hand side exist. But, indeed,
\[
\lim_{h \to 0} h = 0
\]
and
\[
\lim_{h \to 0} o(h) = \lim_{h \to 0} \left( h \frac{o(h)}{h} \right) = \left( \lim_{h \to 0} h \right) \left( \lim_{h \to 0} \frac{o(h)}{h} \right) = 0 \cdot 0 = 0.
\]
So
\[
\lim_{x \to c} f(x) = f(c) + f'(c) \lim_{h \to 0} h + \lim_{h \to 0} o(h) = f(c) + 0 + 0 = f(c)
\]
and thus \( f \) is indeed continuous at the point \( c \).

Problem 4

By definition of derivatives, \( f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \)
This means that for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that we have \( \left| \frac{f(c + h) - f(c)}{h} - f'(c) \right| < \epsilon \) for any \( \epsilon > 0 \).

Now we choose \( \delta \) for \( \epsilon = f'(c) \). (We can do so because \( f'(c) \) is given to be positive.)

So for any \( |h| < \delta \), we have \( \left| \frac{f(c + h) - f(c)}{h} - f'(c) \right| < f'(c) \)

Notice that the inequality is equivalent to: \( -f'(c) < \frac{f(c + h) - f(c)}{h} - f'(c) < f'(c) \)

The left inequality gives \( \frac{f(c + h) - f(c)}{h} > 0 \) for any \( |h| < \delta \).

If \( 0 < h < \delta \), this is equivalent to \( f(c + h) - f(c) > 0 \), or more simply, \( f(c + h) > f(c) \).

Therefore we have \( f(x) > f(c) \) for any \( x \in (c, c + \delta) \).
Problem 5

Consider the function $h$ defined on $[a, b]$ by $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$. Then $h$ is certainly differentiable on $[a, b]$ since it is the product and sum of differentiable functions. Also note that $h(a) = h(b) = 0$. Thus by the mean value theorem there must be a point $c \in (a, b)$ so that $h'(c) = 0$ and this $c$ clearly solves the problem.