1.3.6 Either there exists \( n \) so that \( t_n(x) < t_n(y) \) or \( x = y \). If it is the latter, then since \( y \leq z \) and \( x \) is just the same as \( y \), we know that \( x \leq z \). Similarly, either there exists \( m \) with \( t_m(y) < t_m(z) \) or \( y = z \). If it is the latter, since we already know \( x \leq y \), and \( y \) is just the same as \( z \), we know that \( x \leq z \). Thus it suffices to consider the case with \( t_n(x) < t_n(y) \) and \( t_m(y) < t_m(z) \). Note that when \( t_n(x) < t_n(y) \), it is also the case that \( t_N(x) < t_N(y) \) for any larger \( N \) and similarly for \( y \) and \( z \). Thus, we just pick \( N \) larger than both \( n \) and \( m \). We have \( t_N(x) < t_N(y) \) and \( t_N(y) < t_N(z) \). Since those two inequalities are between rational numbers, we may apply transitivity in the order of rational numbers to get \( t_N(x) < t_N(z) \).

1.3.7 Observe that \( x_n \) is the largest rational number which can be represented with denominator \( 10^n \) which is not an upper bound for \( A \) and \( x_m \) is the largest rational number which can be represented with denominator \( 10^m \). Note that since every fraction with denominator \( 10^n \) is a fraction with denominator \( 10^m \), it must be that \( x_n \leq x_m \). But \( t_n(x_m) \) is the largest fraction with denominator \( 10^n \) which is less than or equal to \( x_m \) and since \( x_m \) is not an upper bound for \( A \), neither is \( t_n(x_m) \). Thus it must be that \( t_n(x_m) = x_n \). Each number \( x_n \) has an expansion

\[
x_n = a_m \ldots a_1.b_1 \ldots b_j \ldots b_N.
\]

Note that because \( t_n(x_m) = x_n \) when \( m > n \), no digit \( a_j \) or \( b_j \) depends on \( n \). Thus there is a real number

\[
x = a_m \ldots a_1.b_1 \ldots b_n \ldots
\]

We claim that \( x \) is an upper bound for \( A \). Note that \( x + 10^{-n} \) is an upper bound for \( A \) for every \( n \) since it is greater than \( x_n + 10^{-n} \). Apply the result of problem 1.4.2 below. Moreover, it is the least upper bound. Suppose not. Then there is an upper bound \( y \) with \( y < x \). This means there is some \( n \), where \( t_n(y) < t_n(x) \) which means that \( t_n(y) + 10^{-n} \) is an upper bound for \( A \) and so is \( x_n \) which is at least as large. This contradicts the definition of \( x_n \).

1.4.2 Suppose \( x < y + \epsilon \) for every \( \epsilon > 0 \). There is an infinite sequence of positions \( n_j \) so that the last digit of \( t_{n_j}(y) \) is not 9. [Otherwise, \( x \) would be a terminating decimal and its \( t_n \)'s would terminate for \( n \) sufficiently large]. Pick \( \epsilon = 10^{-n_j} \). Then for any \( m < n_j \), we have \( t_m(y + 10^{-n_j}) = t_m(y) \). Suppose there is some \( m \) for which \( t_m(y) < t_m(x) \). Then by picking \( n_j > m \) we get a contradiction. Thus either, \( t_m(y) = t_m(x) \) for every \( m \) in which case we have \( x = y \) or there is some \( m \) for which \( t_m(x) < t_m(y) \). Thus we have shown \( x \leq y \).

1.4.3 First, let us show that \( (\sqrt{x})^2 \leq x \). Suppose not. Then \( (\sqrt{x})^2 > x \). Observe that

\[
(\sqrt{x} - \frac{1}{n})^2 = (\sqrt{x})^2 - \frac{2\sqrt{x}}{n} + \frac{1}{n^2}.
\]
By choosing $n$ so large that $(\sqrt{x})^2 - \frac{2\sqrt{x}}{n}$ is still greater than $x$, we see that $\sqrt{x}$ is not the least upper bound. Now let us show that $(\sqrt{x})^2 \geq x$. Suppose not. Then there is some $\epsilon > 0$ so that there are no squares of real numbers between $x - \epsilon$ and $x$. We will contradict this by showing that for every $n$ and every positive real $x$, there is a square between $(1 + \frac{1}{n})^{-2}x$ and $x$. Consider, the two sided sequence of numbers $a_m(n) = (1 + \frac{1}{n})^{2m}$ where $m$ runs over all the integers both positive and negative. By the well ordering principle, there is a largest $m$ so that $a_m(n) < x$. Hence $a_m(n) < x \leq a_{m+1}(x)$. But $a_m(x)$ and $a_{m+1}(x)$ are squares of real numbers which differ by a factor of $(1 + \frac{1}{n})^2$.

1.4.5 Observe that 

$$ (1 + \frac{1}{n})^{\frac{1}{4}} \leq 1 + \frac{1}{4n}, $$

since the first two terms of the binomial expansion of $(1 + \frac{1}{n})^4$ are $1 + \frac{1}{n}$. Thus

$$ |(1 + \frac{1}{n})^{\frac{1}{4}} - 1| \leq \frac{1}{4n}. $$

Choosing $N(\epsilon) = \frac{1}{4\epsilon}$, we see that when $n > N(\epsilon)$,

$$ |(1 + \frac{1}{n})^{\frac{1}{4}} - 1| < \epsilon. $$