Math 108b Problem set 1 due: Thursday January 14th, 2016 4:00 P.M

1. Let $U$ be a bounded open set in $\mathbb{R}^n$. Show there is a constant $C_n$ depending only on $n$, so that $U$ is the union of a countable collection $Q$ of nonoverlapping closed cubes, with the property that for each $Q \in Q$, we have $l(Q) > \frac{d(Q)}{C_n}$, with $l(Q)$ the sidelength of $Q$ and $d(Q)$, the distance between $Q$ and the complement of $U$.

2. Let $f$ be a nonnegative continuous function on an interval $I$ in $\mathbb{R}^n$. Let $\lambda > 0$ be a real number. Show that there is a countable family of nonoverlapping cubes $\{Q_j\}$ with the properties that
\[
\sum_j |Q_j| \leq \frac{(R) \int_I f}{\lambda},
\]
for any $x \in I$ but in the complement of $\cup_j Q_j$, one has that $f(x) \leq \lambda$, and for any $Q_j$, one has
\[
\frac{1}{|Q_j|} (R) \int_{Q_j \cap I} f \leq 2^n \lambda.
\]
(Hint: Consider the set of dyadic cubes $Q$ on which $\frac{1}{|Q|} (R) \int_{Q \cap I} f$ is more than $\lambda$. Which cubes should you pick to be the $Q_j$’s?)

3. Let $f$ be a bounded function on $\mathbb{R}^n$ and $I$ an interval. Suppose that for every $\epsilon > 0$, there is a partition $P$ of $I$ so that
\[
U_P(f) - L_P(f) < \epsilon,
\]
where $U_P(f)$ and $L_P(f)$ are respectively the upper and lower Riemann sums with respect to the partition. Show that $f$ is Riemann integrable on $I$.

4. Let $f$ be a bounded function on $\mathbb{R}^n$. Define
\[
D_f(x) = \limsup_{y \to x} f(y) - \liminf_{y \to x} f(y).
\]
Show that $f$ is continuous at $x$ if and only if $D_f(x) = 0$.

5. Let $f$ be a bounded function on $\mathbb{R}^n$ and $D_f(x)$ be as above. Let $\epsilon > 0$ be a real number. Show that the set $\{x : D_f(x) \geq \epsilon\}$ is closed.