Math 110c Problem Set 2 Due April 28 4:00 P.M.

1. Let \( 1 < p_0, p_1, q_0, q_1 < \infty \). Suppose that \( p_0 \leq q_0 \) and \( p_1 \leq q_1 \). Let \( 0 < \theta < 1 \) and let
\[
\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1},
\]
and
\[
\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}.
\]
Let \( T \) be a sublinear operator on \( L^{p_0} + L^{p_1} \) which is weak type \((p_0, q_0)\) and weak type \((p_1, q_1)\). Show that \( T \) is strong type \((p, q)\).

2. Let \( E \subset \mathbb{R}^n \) be a bounded Lebesgue measurable set of finite measure. Let \( Q \) be a collection of cubes in \( \mathbb{R}^n \) of bounded sidelength which covers \( E \). Show that there is a countable subcollection \( \{Q_j\} \) of \( Q \) which is pairwise disjoint and so that \( \{5Q_j\} \) covers \( E \). (Here, for any cube \( Q \), we define \( 5Q \) to be the cube having the same center as \( Q \) and 5 times the sidelength.) Hint: Take \( Q_1 \) to be a cube at least half as long as the largest in \( Q \). Then remove from \( Q \) all cubes which intersect \( Q_1 \).

3. Use problem 2 to give a proof of the weak type \((1,1)\) boundedness of \( M' \), the centered cubical maximal function on \( \mathbb{R}^n \).

4. Let \( f \) be a real valued function on \([0, 1]\) whose derivative exists and is continuous at every point of \([0, 1]\). Then the graph of \( f \)
\[
G = \{(x, f(x)) : x \in [0, 1]\}
\]
is a rectifiable curve. Suppose that the length of \( G \) is equal to \( M \). Let \( K > 0 \) be a real number. Show that there is a measurable set \( E \subset [0, 1] \) with
\[
|E| \leq \frac{2M}{K},
\]
so that for any \( x, y \in [0, 1] \), with neither \( x \) nor \( y \) in \( E \), we have the estimate:
\[
|f(x) - f(y)| \leq K|x - y|.
\]
Hint: Apply the weak type \((1,1)\) bound on the 1-dimensional Hardy Littlewood maximal operator given in section 2.4 of Duoandikoetxea.

5. Let \( E \subset [0, 1] \) be a measurable set with \( |E| = \epsilon > 0 \). Let
\[
f(x) = \chi_E(x).
\]
Define the minimal function
\[
(m_1 f)(x) = \inf_{r < 1} \frac{1}{r} \int_{I_r(x)} |f(x)| \, dx,
\]
where $I_r(x)$ is the interval centered around $x$ with length $r$. (This differs from the Hardy Littlewood maximal function in that the sup becomes an inf and the intervals are always chosen with length less than 1.) Define

$$E_{bad} = \{x \in E : M_1 f(x) < \frac{e}{100}\}.$$

Show that

$$|E_{bad}| \leq \frac{e}{10}.$$  

(In other words most points of $E$ are good.) Hint: Cover $E_{bad}$ by intervals in whose five-fold product, the set $E$ is sparse. Then use the idea of problem 2.