1. (10 points) Using the power series definition of $e^z$ and induction on its terms, show that

$$e^z e^w = e^{z+w},$$

for $z$ and $w$ complex numbers.

2. (10 points) Let $z$ and $w$ be complex numbers. Show directly from the algebraic definition of magnitude that

$$|z + w| \leq |z| + |w|.$$

3. (10 points) Find identities for $\sin 5\theta$ and $\cos 5\theta$ strictly in terms of $\sin \theta$ and $\cos \theta$.

4. (10 points) Let $f(t)$ be a once continuously differentiable function from the reals to the complex numbers which is 1-periodic. That is let $f(t+1) = f(t)$ for all real $t$. Suppose that $f(t) \neq 0$ for any real $t$. Consider

$$\int_0^1 \frac{f'(t)}{f(t)} dt.$$

Since the integrand is continuous, this integral can be defined as

$$\int_0^1 \frac{f'(t)}{f(t)} dt = \int_0^1 \text{Re} \left( \frac{f'(t)}{f(t)} \right) dt + i \int_0^1 \text{Im} \left( \frac{f'(t)}{f(t)} \right) dt.$$

Show that the integral is an integer multiple of $2\pi i$.

5. (10 points) Use Problem 4 to give a proof of the fundamental theorem of algebra. Namely any degree $n$ polynomial of the complex numbers with $n > 0$ has a complex root. (See hints page.)