1 Problem 3.3.1

Since $f$ is $O(1)$, there are $\epsilon > 0$ and a constant $C$ so that if $|h| < \epsilon$ then $f(h) < C$. Since the definition of limit as $h \to 0$ only depends on values of $h$ smaller than $\epsilon$, we have

$$ \lim_{h \to 0} \frac{f(h)g(h)}{h} \leq \lim_{h \to 0} \frac{Cg(h)}{h} = C \lim_{h \to 0} \frac{g(h)}{h} = 0 $$

since $g$ is $o(h)$.

2 Problem 3.3.2

Write $f'(c) = 1 + r$ for $r > 0$. Find $\delta > 0$ so that if $|x - c| < \delta$ then

$$ \left| \frac{f(x) - f(c)}{x - c} - (1 + r) \right| < \frac{r^2}{2}. $$

Then for $x \in (c, c + \delta)$ we have $f(x) - f(c) \geq x - c$.

3 Problem 3.3.3

We prove this problem by induction on $n$. So we want to show that if $f$ and $g$ are two functions both $n$ times differentiable, then $fg$ is also $n$ times differentiable and $(fg)^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x)g^{(n-j)}(x)$.

If $n = 1$, then we know that $f$ and $g$ are both differentiable, so we know that $fg$ is also differentiable, and

$$(fg)^{(1)}(x) = f(x)g'(x) + f'(x)g(x) = f^{(0)}(x)g^{(1)}(x) + f^{(1)}(x)g^{(0)}(x) = \sum_{j=0}^{1} \binom{1}{j} f^{(j)}(x)g^{(1-j)}(x).$$

So when $n = 1$ the problem is correct.

Now assume that the problem is true for $n - 1$, we want to show it is true for $n$. So we assume $f$ and $g$ are $n$ times differentiable. When $f$ and $g$ are $n$ times differentiable, in particular they are $n - 1$ time differentiable, so by the induction hypothesis $fg$ is $n - 1$ times differentiable, and $(fg)^{(n-1)}(x) = \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)}(x)g^{(n-1-j)}(x)$.

In order to show that $fg$ is $n$ times differentiable, it’s enough to show that $(fg)^{(n-1)}$ is differentiable. So we need to show that $\sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)}(x)g^{(n-1-j)}(x)$ is differentiable. But since $f$ and $g$ are both $n$ times differentiable, for $j = 0, \ldots, n - 1$ we know that $f^{(j)}$ and $g^{(n-1-j)}$ are both differentiable, so their product, $f^{(j)}g^{(n-1-j)}$ is also differentiable. We also know that a linear combination of differentiable functions is differentiable, so we get that $(fg)^{(n-1)}(x) = \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)}(x)g^{(n-1-j)}(x)$ is differentiable.
Now that we know \(fg\) is \(n\) times differentiable, we just need to show that \((fg)^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x)g^{(n-j)}(x)\).

\[
(fg)^{(n)}(x) = \frac{d}{dx} (fg)^{(n-1)} = \frac{d}{dx} \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)}(x)g^{(n-1-j)}(x)
\]

\[
= \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{d}{dx} (f^{(j)}(x)g^{(n-1-j)}(x))
\]

\[
= \sum_{j=0}^{n-1} \binom{n-1}{j} \left( \frac{d}{dx} (f^{(j)}(x))g^{(n-1-j)}(x) + f^{(j)}(x) \frac{d}{dx} (g^{(n-1-j)}(x)) \right)
\]

\[
= \sum_{j=0}^{n-1} \binom{n-1}{j} \left( f^{(j+1)}(x)g^{(n-1-j)}(x) + f^{(j)}(x)g^{(n-j)}(x) \right)
\]

\[
= \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j+1)}(x)g^{(n-j)}(x) + \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)}(x)g^{(n-j)}(x)
\]

\[
= \sum_{j=1}^{n} \binom{n-1}{j-1} f^{(j)}(x)g^{(n-j)}(x) + \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)}(x)g^{(n-j)}(x)
\]

\[
= f^{(0)}(x)g^{(n)}(x) + \sum_{j=1}^{n} \binom{n-1}{j-1} f^{(j)}(x)g^{(n-j)}(x) + f^{(n)}(x)g^{(0)}(x)
\]

\[
= f^{(0)}(x)g^{(n)}(x) + \sum_{j=1}^{n} \binom{n}{j} f^{(j)}(x)g^{(n-j)}(x) + f^{(n)}(x)g^{(0)}(x)
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x)g^{(n-j)}(x)
\]

4 Problem 3.5.1

A function \(f\) is convex on \([a, b]\) if and only if for any \(x_1 < x_2 < x_3 \in [a, b]\) we have \(\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}\). Choose \(c < d \in [a, b]\). For any \(x < x' \in (c, d)\) we have \(\frac{f(x) - f(c)}{x - c} \leq \frac{f(d) - f(x')}{d - x'}\). Taking the limit as \(x \to c\) and \(x' \to d\) we see \(f'(c) \leq f'(d)\).
5 Problem 3.5.2

Suppose toward a contradiction that $c < d$ are both local minima. Assume first that $f(c) \leq f(d)$. Find $d - c > \delta > 0$ so that for all $x \in (d - \delta, d)$ we have $f(x) \geq f(d)$. Since $f$ is not constant on $(d - \delta, d)$ we can find $y \in (d - \delta, d)$ so that $f(y) > f(d)$. Writing $y = tc + (1-t)d$ for $0 < t < 1$ we see $f(y) = f(tc + (1-t)d) > f(d) \geq tf(c) + (1-t)f(d)$ contradicting convexity of $f$. The case $f(c) \geq f(d)$ can be proved in the same way.