1 Problem 2.1.5

Use the squeeze theorem to calculate
\[ \lim_{n \to \infty} n \left( \sqrt{4 + \frac{3}{n}} - 2 \right). \]

Solution. Let
\[ a_n = n \left( \sqrt{4 + \frac{3}{n}} - 2 \right). \]
We have
\[ \left(2 + \frac{3}{4n}\right)^2 \geq 4 + \frac{3}{n} \]
and thus
\[ 2 + \frac{3}{4n} \geq \sqrt{2 + \frac{3}{n}}. \]
Hence if we define
\[ b_n = n \left(2 + \frac{3}{4n} - 2\right) = \frac{3}{4} \]
then \(a_n \leq b_n\) for all \(n\). Similarly, we have
\[ \left(2 + \frac{3}{4n} - \frac{1}{n^2}\right)^2 \leq 4 + \frac{3}{n} \]
and so if we define
\[ c_n = n \left(2 + \frac{3}{4n} - \frac{1}{n^2} - 2\right) = \frac{3}{4} - \frac{1}{n} \]
then \(c_n \leq a_n\) for all \(n\). Clearly \(\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = \frac{3}{4}\) so by the squeeze theorem we have \(\lim_{n \to \infty} a_n = \frac{3}{4}\).

2 Problem 2.2.3

Prove the following series converges:
\[ \sum_{n=1}^{\infty} \frac{n^n}{(n!)^3}. \]

Solution. By pairing the terms in the product defining \(n!\) from the outside in, we see that \(n! \geq \frac{1}{2} n^\frac{n}{2}\). Thus
\[ \sum_{n=1}^{\infty} \frac{n^n}{(n!)^3} \leq 8 \sum_{n=1}^{\infty} \frac{n^n}{n^\frac{n}{2}} \]
\[ = 8 \sum_{n=1}^{\infty} \frac{1}{n^\frac{n}{2}} \]
\[ \leq 8 \sum_{n=1}^{\infty} \frac{1}{n^2} \]
and the last series above is known to converge.
3 Problem 2.3.4

Let \( (a_n)_{n=1}^\infty \) be a sequence of real numbers. Define \( B_n = \{ a_k : k \geq n \} \) and \( b_n = \sup B_n \). Prove the sequence \((b_n)\) is decreasing. Define \( \limsup_{n \to \infty} a_n = \lim_{n \to \infty} b_n \). Prove there is a subsequence of \((a_n)\) which converges to \( \limsup_{n \to \infty} a_n \).

**Solution.** Note that if \( A \) and \( B \) are sets of real numbers and \( A \subseteq B \) then \( \sup A \leq \sup B \). Since \( B_{n+1} \subseteq B_n \) we see that \( b_{n+1} \leq b_n \).

Write \( L = \limsup_{n \to \infty} a_n \). Given \( k \in \mathbb{N} \) find \( l \geq k \) so that \( |L - b_l| < \frac{1}{2k} \). Then find \( a_{n_k} \in B_l \) with \( |b_l - a_{n_k}| < \frac{1}{2k} \).

By the triangle inequality we have \( |a_{n_k} - L| < \frac{1}{k} \). Thus the sequence \((a_{n_k})_{k=1}^\infty\) converges to \( L \).

4 Problem 2.3.5

Let \( L \) be the limit of some subsequence of the \( a_n \). Show \( L = \limsup_{n \to \infty} a_n = \lim_{N \to \infty} b_N \).

**Solution.** Suppose toward a contradiction that \((a_{n_k})_{k=1}^\infty\) converges to some \( L \) with \( L > \limsup_{n \to \infty} a_n \). Write \( L = \limsup_{n \to \infty} a_n + c \) for some \( c > 0 \). Given \( N \in \mathbb{N} \), find \( k \geq N \) such that \( |a_{n_k} - L| < \frac{c}{2} \). This implies \( a_{n_k} \geq L - \frac{c}{2} \). Since \( a_{n_k} \in B_N \) it follows that

\[
b_N = \sup B_N \geq a_{n_k} \geq L - \frac{c}{2}.
\]

Thus \( \lim_{N \to \infty} b_N \geq L - \frac{c}{2} \), which contradicts our assumption that \( L = \limsup_{n \to \infty} a_n + c \).

5 Problem 2.3.6

Let \((a_n)\) be a positive sequence of real numbers. Suppose that

\[
L = \limsup_{n \to \infty} a_n^{\frac{1}{n}}
\]

is nonzero and finite. Show that \( \frac{1}{L} \) is the radius of convergence of the power series \( \sum_{n=1}^\infty a_n x^n \).

**Solution.** By comparison with a geometric series, we see that if \( 0 \leq b_n^{\frac{1}{n}} < c < 1 \) for all \( n \) then \( \sum_{n=1}^\infty b_n \) converges. Suppose \( |x| < \frac{1}{L} \), say \( |x| = \frac{r}{L} \) for some \( r < 1 \). Choose any \( c \) with \( r < c < 1 \), then if \( n \) is large enough we will have \( a_n^{\frac{1}{n}} |x| < c \) so that \((a_n x^n)^{\frac{1}{n}} < c \) and \( \sum_{n=1}^\infty a_n x^n \) converges by our previous criterion.
Now suppose $|x| > \frac{1}{L}$. Choose a subsequence $(a_{n_k})$ with $\lim_{k \to \infty} \frac{1}{a_{n_k}} = L$. Then $\lim_{k \to \infty} \frac{1}{a_{n_k}} |x| = \lim_{k \to \infty} (a_{n_k} x^{n_k}) \frac{1}{n_k} > 1$, so the terms of $\sum_{n=1}^{\infty} a_n x^n$ do not approach zero and hence the series does not converge.