Spectral Packing Dimensions of 1-dimensional quasi-periodic Schrödinger operators

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Main Object: Quasi periodic Schrödinger operator on $l^2(\mathbb{Z})$:

$$(Hu)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}, \quad \theta \in \mathbb{T}, \quad \alpha \in \mathbb{R}\setminus\mathbb{Q} \quad (1)$$

In this talk, we discuss the packing dimension of the spectral measure of the above operator.

1. Preliminary and Main result on packing dimension
2. Outline of the Proof
3. Related model and Remaining problems
For any subset $S$ of $\mathbb{R}$ and $\gamma \in [0, 1],$

<table>
<thead>
<tr>
<th>Hausdorff measure and dimension</th>
<th>packing measure and dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$-cover: $\bigcup_{i=1}^{\infty} b_i \supseteq S$</td>
<td>$\delta$-packing: $\bigcup_{i=1}^{\infty} b_i$, disj. centered in $S$</td>
</tr>
<tr>
<td>$h^\gamma(S) = \lim_{\delta} \inf_{\delta-c} \sum_{i=1}^{\infty}</td>
<td>b_i</td>
</tr>
<tr>
<td>$\dim_H(S) = \sup{\gamma : h^\gamma(S) = \infty}$</td>
<td>$P^\gamma(S) = \inf \left{ \sum_{k=1}^{\infty} P^\gamma(S_k) : S \subset \bigcup_{k=1}^{\infty} S_k, S_k \text{ disj.} \right}$</td>
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Given $\gamma \in [0, 1]$, a Borel measure $\mu$ on $\mathbb{R}$, $\star$ will represent Hausdorff or packing respectively. $\mu$ is called:

- $\gamma$-\star continuous: $\forall S, h^\gamma(S) = 0 \Rightarrow \mu(S) = 0$ (resp. $P^\gamma(S) = 0$).
- $\gamma$-\star singular: supported on $S$ with $h^\gamma(S) = 0$ (resp. $P^\gamma(S) = 0$).
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**Definition**

A Borel measure $\mu$ on $\mathbb{R}$ is said to have exact $\star$ dimension $\gamma$, denoted by $\dim_H(\mu)$ or $\dim_P(\mu)$, if for any $\epsilon > 0$, it is simultaneously $(\gamma - \epsilon)$-$\star$ continuous and $(\gamma + \epsilon)$-$\star$ singular.
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- zero-$\star$ dimensional: $\gamma$-$\star$ singular for every $\gamma > 0$
- one-$\star$ dimensional: $\gamma$-$\star$ continuous for every $\gamma < 1$. 

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Almost Mathieu operator

\[(H_{\lambda, \theta, \alpha} u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n, \, \lambda > 0.\]  (2)
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‘Trivial’ region: pure a.c. or pure point spectrum appears:

- $0 < \lambda < 1$, $\forall \theta, \alpha$, $\mu = \mu_{ac}$, $\dim_H(\mu) = \dim_P(\mu) = 1$
- $\lambda > 1$, $\alpha \in D.C.$, $\theta$ non-resonant, $\mu = \mu_{pp}$, $\dim_H(\mu) = \dim_P(\mu) = 0$
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Interesting region: singular continuous spectrum appears.

- Jitomirskaya, Last: \(\lambda > 1, \forall \alpha \in \mathbb{R}\setminus\mathbb{Q}, \forall \theta, \dim_H(\mu) = 0;\)
- Simon: generalized to Jacobi operator with positive Lyapunov exponent.
- Last, Shamis: \(\lambda = 1, \exists\) dense \(G_\delta\) set of \(\alpha, \forall \theta, \dim_H(\sigma(H_\theta)) = 0.\)
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Problem

What about the packing dimension of spectral measure of AMO?
Consider quasi periodic Schrödinger operator on $l^2(\mathbb{Z})$:

$$ (Hu)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}, \quad \theta, \alpha \in \mathbb{T} $$

(3)

where $V : \mathbb{T} \rightarrow \mathbb{R}$ is assumed to be Lipschitz continuous.

Denote by

$$ \beta(\alpha) := \limsup_n \frac{\log q_{n+1}}{q_n} $$

where $\frac{p_n}{q_n}$ is the $n^{th}$ rational approximation of $\alpha$. 

Theorem (Main) If $\beta(\alpha) = \infty$, for any $\theta$, the spectral measure $\mu_\theta$ of Schrödinger operator (3) has packing dimension one.
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Corollary on AMO

AMO:

\[(H_{\lambda,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n\]
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Corollary

*If* $\beta(\alpha) = \infty$, *then for any* $\lambda > 0$ *and* $\theta$, *dim*$_P(\mu_\theta) = 1$. 
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Corollary

If \(\beta(\alpha) = \infty\), then for any \(\lambda > 0\) and \(\theta\), \(\dim_P(\mu_\theta) = 1\).

Let \(dN\) be the density states measure and \(\Sigma\) be the spectrum.

Corollary

If \(\lambda = 1, \beta(\alpha) = \infty\), then \(\dim_P(\Sigma) = \dim_P(dN) = 1\).
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If \( \lambda = 1, \beta(\alpha) = \infty \), then \( \dim_P(\Sigma) = \dim_P(dN) = 1 \).

- \( \lambda > 1, \beta = \infty, \forall \theta, 0 = \dim_H(\mu_\theta) < \dim_P(\mu_\theta) = 1 \).
- \( \lambda = 1, \exists \text{ dense } G_\delta \text{ set of } \alpha, 0 = \dim_H(\Sigma_\alpha) < \dim_P(\Sigma_\alpha) = 1 \).
\[ Hu = Eu \iff A_n(\theta, E) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} \]

\[ A(\theta, E) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \quad A_n = \prod_n A(\theta + j\alpha) \]
Rational approximation and Bound for trace map

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Let \( \Lambda := \sup_{E, \theta} \log \| A(\theta, E) \| \). Let \( q_{n_k} \) be the subsequence of \( q_n \) such that \( \frac{\log q_{n_k+1}}{q_{n_k}} > \beta - \Lambda/200 \) (denote by \( q_k \) for simplicity).
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**Theorem**

If \( \beta > 30\Lambda \), then for any \( \theta \) and for \( \mu \) a.e. \( E \), there is \( K(E) \) such that

\[ |\text{Trace}(A_{q_k}(E, \alpha, \theta))| < 2 - e^{-10\Lambda q_k}, \quad k \geq K(E). \]
Conclusion of the Main Theorem

Let $M : \mathbb{C}^+ \leftrightarrow \mathbb{C}^+$ be the m-function of $H$ associated with spectral measure $\mu$. Combine the previous estimates on trace with subordination theory developed by Jitmoskaya and Last.

There is some absolute constant $C_0$, for any $\gamma < 1$, if $\beta > C_0 1 - \gamma \Lambda$ then spectrally a.e. $E$, there is a sequence $\eta_k \to 0$, such that $\eta_1 - \gamma k |M(E + i \eta_k)| < 100$, $k \geq K(E)$ (5)

Therefore, $D_\gamma \mu(E) < \infty \mu$-a.e. $E$, i.e., the spectral measure $\mu$ is $\gamma$-packing continuous.

$\beta = \infty$: $\mu$ is $\gamma$-packing continuous for any $\gamma < 1$ $\Rightarrow$ $\dim P(\mu) = 1$.

$\beta > C_0 \Lambda$: $\exists \gamma_0 > 0$ s.t. $\mu$ is $\gamma$-packing continuous for any $\gamma < \gamma_0$ $\Rightarrow$ $\dim P(\mu) \geq \gamma_0 > 0$. 
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**Theorem**

There is some absolute constant \( C_0 \), for any \( \gamma < 1 \), if \( \beta > \frac{C_0}{1-\gamma} \Lambda \) then spectrally a.e. \( E \), there is a sequence \( \eta_k \to 0 \), such that

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- $\beta > C_0 \Lambda$: $\exists \gamma_0 > 0$ s.t. $\mu$ is $\gamma$-packing continuous for any $\gamma < \gamma_0$ \Rightarrow $\dim_P(\mu) \geq \gamma_0 > 0$
Application to dynamical exponents

\[
\langle |X|^{p}_{\delta_0}(T) \rangle = \frac{2}{T} \int_{0}^{\infty} e^{-2t/T} \sum_{n} |n|^p |\langle e^{-itH}\delta_0, \delta_n \rangle|^2 dt, \quad p > 0.
\]

\[
DE_{\delta_0}^+(p) = \limsup_{T \to \infty} \frac{\log \langle |X|^{p}_{\delta_0}(T) \rangle}{p \log T}, \quad DE_{\delta_0}^+(p) \geq dim_P(\mu)
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**AMO, \( \lambda > 1 \):**

- Last: \( \exists \alpha = \alpha(\lambda, \theta) \) such that \( DE_{\delta_0}(2) = 1. \)
- Damanik, Tcheremchantsev: \( \beta(\alpha) = 0, \ DE_{\delta_0}^+(p) = 0, \ \forall p > 0. \)
Application to dynamical exponents

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- Damanik, Tcheremchantsev: \( \beta(\alpha) = 0, \ DE^+_{\delta_0}(p) = 0, \forall p > 0 \).

Corollary

If \( \beta(\alpha) = \infty \), the dynamics of the Schrödinger operator \( H \) in (3) is quasi-ballistic, namely, \( DE^+_{\delta_0}(p) = 1 \).
Sturm Hamiltonian:

\[(Hu)_n = u_{n+1} + u_{n-1} + \lambda \chi_{[1-\alpha,1)}(n\alpha + \theta \mod 1)u_n, \; \alpha \in \mathbb{R}\setminus\mathbb{Q}\]
Application to Sturm Hamiltonian

Sturm Hamiltonian:

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**Theorem**

If \(\beta(\alpha) = \infty\), there is set \(\Theta \subseteq [0,1]\) with full Lebesgue measure, such that for any phase \(\theta \in \Theta\) and for any coupling constant \(\lambda > 0\), the spectral measure \(\mu_\theta\) of Sturm Hamiltonian \(H_{\theta,\lambda,\alpha}\) has packing dimension one.

**Corollary**

If \(\beta(\alpha) = \infty\), the packing dimension of the density states of measure \(dN_{\lambda,\alpha}\) and the packing dimension of the spectrum \(\Sigma_{\lambda,\alpha}\) are both equal to one.
Fibonacci Hamiltonian, $\alpha = \frac{\sqrt{5} - 1}{2}$, Damanik, Gorodetski, Yessen:

$$\dim_H(dN_\lambda) = \dim_P(dN_\lambda) < \dim_H(\Sigma_\lambda)$$
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- S.H., Wen, Liu: for $\lambda > 20$, $\exists \Omega$ with zero Leb measure, s.t., $\dim_H(\Sigma_{\lambda,\alpha}) = 1$ iff $\alpha \in \Omega$
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\[
\dim_H(\Sigma_{\lambda,\alpha}) = 1 \iff \alpha \in \Omega
\]

\( \exists \alpha \notin \Omega \) while \( \beta(\alpha) = \infty \)

**Corollary**

*There are frequencies \( \alpha \) such that \( \dim_H \Sigma_{\alpha,\lambda} < \dim_P \Sigma_{\alpha,\lambda} = 1 \)*
Remaining problems: dimension transition for AMO

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(Hausdorff, packing, box) dimension for general \( \alpha \)
J. Bellissard conjectured: $\dim_H(\Sigma_\alpha) = \gamma(\alpha) \in (0, 1/2]$ for almost every $\alpha$ and $\lambda = 1$.

Numerical results: $\dim_B(\Sigma_\alpha) = 1/2$ almost every $\alpha$ and $\lambda = 1$. 

Remaining problems: dimension transition for AMO

$\beta = C_0 \log \lambda$: is there a transition line?

$\forall \theta$, zero dimension?

$p = \log \lambda$:

$\beta = \log \lambda$

Resonant $\theta$, zero dimension?

Packing singular? Exact packing dimension?
Thank you!