Macroeconomic models with Heterogeneous Agents

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February 17, 2015
Outline of the talk

- Prehistory of macroeconomics
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- Lucas’s critique and dynamic programming
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- Krussel-Smith’s heterogeneous agents model
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- Prehistory of macroeconomics
- Lucas’s critique and dynamic programming
- Krussel-Smith’s heterogeneous agents model
- A return to mathematics
Prehistory of Macroeconomics

- Microeconomics: the law of supply and demand
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- Demand curve, supply curve, auctions
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- Multiple goods and concavity of indifference curves. (Butter and margarine)
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- Demand curve, supply curve, auctions
- Multiple goods and concavity of indifference curves. (Butter and margarine)
- Quantity theory of money
Hick’s classical theory of interest rates and employment

- Machinery fixed. A wage $w$ fixed. Labor can produce either investment goods or consumer goods. Investment goods produced $x = f(N_x)$ with $f$ a function involving available machinery and $N_x$ the labor expended on investment goods. Similarly $y = g(N_y)$ with $y$ consumer goods produced and $N_y$, the labor expended on consumer goods.
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The price of consumption goods is $w \frac{dN_x}{dx}$ and the price of investment goods is $w \frac{dN_y}{dy}$. Total income is given by

$$I = wx \frac{dN_x}{dx} + wy \frac{dN_y}{dy}.$$
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▶ Hicks calls this the Cambridge quantity equation: (He was an Oxford man!)

$$M = kI.$$ 

Here $M$ is total supply of money.
Hick’s Classical theory, cont.

- $I_x = C(i)$, namely the rate of return depends on how much is invested. On the other hand, $C(i) = S(i, I)$, how much will be invested depends on the rate of return.

Criticism: the relationship of money supply to income is arbitrary.
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Keynes’ Special theory of interest rates and employment

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- Predictions can be made of the result of shocks to the functions \( C \) and \( S \) and \( L \). Is this any way to do macroeconomics?
Lucas’ critique

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- When a central banker manipulates the functions of one of these supply/demand models, it might not achieve the desired effect, because people will anticipate the action.
- People aren’t stupid. In fact, they know the complete stochastic properties of the universe. (At least if they live inside a rational expectations model.)
Example in rational expectations: Neoclassical growth model

- A single, infinitely-lived agent (the representative agent) must make a decision how much to invest and how much to consume in each discrete time period.

\[ k_1 \]

\[ c_1 \]

\[ 0 \leq c_1 \leq k_1 \]

\[ k_{j+1} = k_j - c_j + \alpha \]

\[ 0 < \alpha < 1 \]

\[ u \]

\[ \beta \]

\[ \beta < 1 \]

\[ \sum_{j=1}^{\infty} \beta^{j-1} u(c_j) \]
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- At the $j + 1$st period, the agent will have wealth, $(k_j - c_j)^\alpha$, the output of a Cobb Douglas machine. Here $0 < \alpha < 1$.
- The agent has a utility function $u$ which is concave and increasing with infinite derivative at 0. He has a discounting rate $\beta < 1$. His goal is to optimize

$$\sum_{j=1}^{\infty} \beta^{j-1} u(c_j).$$
Dynamic programming solves Neoclassical growth model

- We define $V(k)$, the value function at $k$ to be the optimal value of the sum

$$\sum_{j=1}^{\infty} \beta^{j-1} u(c_j),$$

when $k_1 = k$. 

- Then the agent just has to choose $c_1$ so as to maximize $u(c_1) + \beta V(k - c_1)$. The only problem is we don't know there's a function $V$. 

- We introduce $V_0$, a guess for $V$ which is increasing and concave and has infinite derivative at 0. There is unique $c_1$ to optimize $u(c_1) + \beta V_0(k - c_1)$ and define the maximum to be $V_1(k)$. 

- We observe that $V_1$ is concave, increasing, and has infinite derivative at 0, and we iterate. Eventually the process converges.
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What has Lucas gained?

- The model we’ve just presented is deterministic. But it wasn’t essential. The model could be subject to shocks. For instance, the payoff to investing \( k - c \) in \( j \)th period could be \( T_j(k - c)^\alpha \) with \( T_j \) a random variable. It helps if \( T_j \) is independent, identically distributed. \( T_j \) is called the total productivity factor.
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- The agent then optimizes

\[ u(c_1) + E(\sum_{j=2}^{\infty} \beta^{j-1} u(c_j)), \]

using dynamic programming.

To get the results you want from an economic model, add the features you care about. You can build a representative agent model to do almost anything you want.

Lucas’ school considers these models to give microeconomic foundations to macroeconomics.

These models have good mathematical properties. It usually isn’t hard to prove a value function exists.
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Criticism of representative agents

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- It’s really hard to model distribution of wealth in a representative agent model.
- True microeconomic foundations for macro models should include a true microeconomy.
Krussel-Smith model

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- The entire economy produces $I^\alpha$ output. Of this, $\alpha I^\alpha$ is distributed to the investors in proportion to their investments. The remaining $(1 - \alpha)I^\alpha$ is distributed evenly to employed workers.
Krussel-Smith results

- This model is somewhat more complicated than the representative agent model. The actions of any agent depend on what the others are doing and thus on the distribution of wealth.

- Krussel and Smith were not able to prove that this model has a unique solution.

- However, they were able to solve the model numerically.

- Moreover, they observed that they only needed a few moments of the distribution of wealth to get a good approximation to their solution. Most agents were behaving the same. They referred to this phenomenon as approximate aggregation.
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Introducing the auctioneer

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- We introduce a new figure into our model whose job is to do the real hard calculations. She is the auctioneer. Her job is to announce the aggregate investment in each time period (for each resolution of shocks.)
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- There is only one constraint on her predictions. They have to be chosen so as to be guaranteed to come true. Finding that choice of predictions is the auctioneers problem. She knows if there is a unique solution. We don’t.
Agent’s problem

Once the auctioneer has done her work, all the agent has to do is believe her. He is knows in any future state $\sigma$ at time period $j$ what will be his return on investment $r_\sigma$. If he invests $(k_j(x) - c_j(x))dx$, he will receive back $r_\sigma(k_j(x) - c_j(x))dx$. We omit the log $dx$'s because they don’t play a role in the optimization. It’s a renormalization, if you will. The agent is not as smart as the auctioneer. He need only solve a problem that we know how to solve by dynamic programming.
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Approximate Aggregation and the agent’s problem

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▶ The more capital that an agent has, the less will he be affected by the employment shock, thus as the wealth of an agent moves higher and higher above the mean, the larger is the interval of possible wealth on which the proportion invested approximately does not change.

▶ The share invested by poor agents can vary faster but they contribute very little to aggregate investment.
Theorem: In an $N$ agent version of the Krussel-Smith model, for any $\epsilon > 0$ there is an integer $M = O\left(\frac{1}{\epsilon}\right)$ so that we may divide our agents into $M$ sets $B_1, \ldots, B_M$ and assign each set a number $r_j$ so that if the $l$th agent has capital $k_l$ and invests $i_l$ and if the aggregate investment is $I$ then

$$\sum_{j=1}^{M} \sum_{l \in B_j} \left| \frac{r_j k_l - i_l}{I} \right| \leq \epsilon.$$
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▶ Similar stronger estimates can be obtained about the approximation of rate of investment by higher degree polynomials.

▶ This goes a long way towards explaining Krussel and Smith’s approximate aggregation.
Numerics for the agents’ problem

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- Given the auctioneer’s information, solving the agents problem at different wealth levels admits considerable parallelism and is naturally a job for GPU programming.
- A simple algorithm for finding both the investment and value function is the agent’s by backwards induction is to approximate these functions at one time period by polynomials of degree $2m – 1$ on equal length intervals in logspace and then solving for the functions on the next period at $m$ points around each interval and fitting a new polynomial to the curve.
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- This procedure also admits considerable parallelism.
In the continuum model, the auctioneer must only announce one aggregate per time period. This means we can hope to do numerics for the auctioneer’s problem without any “curse of dimensionality.”

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We are still in the debugging stage so cannot report how practical this procedure is.
Steady states for Krussel Smith

- Much current numerical work for Krussel Smith centers around perturbing off steady states. Numerical simulations tend to approach steady states fairly quickly.

- Steady states are much easier to understand than general states. Because the state is steady, the auctioneer announces the same number $I$ for each period.

- One can imagine that an approach to finding steady states is to find for each choice of $I$, a distribution of wealth which is steady under the agent's problem for that $I$, let's call it $k_I(x) dx$.

- By abuse of notation, we let $I$ be the aggregate investment generated from distribution $k_I$ and the auctioneer's announcement of $I$. Then shifting $I$, hopefully we have just a one good supply/demand problem. Unfortunately $k_I$ is highly nonunique without a more complicated employment shock.
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Open problems

- How do you solve the auctioneer’s problem?
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▶ How should you really model changes in distribution of wealth?
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- How do you solve the auctioneer’s problem?
- How should you really model changes in distribution of wealth?
- Can we see why some remedies are better than others. (For instance, Piketty proposes both wealth taxes and confiscatory income taxes. Is one of the two enough? How do their effects differ?)
Goodbye!

- Thanks for listening!