Energy-critical NLS with potentials of quadratic growth

Casey Jao
UCLA
17 Feb 2015
33rd Western States Meeting
The semilinear Schrödinger equation

\[ i \partial_t u = (-\frac{1}{2} \Delta + V)u + \mu |u|^p u, \quad u(0, x) = u_0(x) \]

- \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \ V = V(x) \) real potential, \( p > 0, \mu = \pm 1 \).
- \( \mu = +1 \) “defocusing”; \( \mu = -1 \) “focusing.”
- Physics: water waves, fiber optics, BEC.

Conservation laws:

\[ M[u(t)] := \int_{\mathbb{R}^d} |u(t)|^2 \, dx = M[u_0] \]

\[ E[u(t)] := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 + V|u|^2 + \frac{2\mu}{p+2}|u|^{p+2} \, dx = E[u_0] \]
Symmetries and criticality

\[ i \partial_t u = -\frac{1}{2} \Delta u + |u|^p u, \quad u(0) \in H^1(\mathbb{R}^d) \]

\[ E[u] = \int \frac{1}{2} |\nabla u|^2 + \frac{2}{p+2} |u|^{p+2} \, dx \]

- Scaling \( u^\lambda(t, x) = \lambda^{-\frac{2}{p}} u(\lambda^{-2} t, \lambda^{-1} x) \), \( E[u^\lambda] = \lambda^{d-2-\frac{4}{p}} E[u] \).
- \( p < \frac{4}{d-2} \) subcritical, \( p = \frac{4}{d-2} \) critical (everything scale-invariant): eqn, energy, Strichartz spacetime norms \( \|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \).
- Local theory:
  - Subcritical: LWP with lifespan \( T = C(M(u_0), E(u_0)) \). GWP from conservation laws (Ginibre-Velo '85, Kato '87, Tsutsumi '87, Cazenave-Weissler '88).
  - Critical: LWP with lifespan \( T = C(u_0) \) (Cazenave-Weissler '91). GWP much harder.
- Some key players in global theory for \( p = \frac{4}{d-2} \) ('98-'06): Bourgain, Keraani, CKSTT, Ryckman, Visan, Kenig-Merle.
Quadratic potentials

- $V \neq 0$ breaks symmetries; Fourier analysis less useful.
- $V = \frac{1}{2}x^2$ most accessible nontrivial case; exact formula for $e^{it(\frac{1}{2}\Delta - V)}$.

$$i\partial_t u = (-\frac{1}{2}\Delta + \frac{1}{2}x^2)u + |u|^p u, \quad u(0) \in \Sigma$$

$$\|f\|_\Sigma^2 = \|\nabla f\|_{L^2}^2 + \|xf\|_{L^2}^2$$

$$E[u] = \int \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|xu|^2 + \frac{2}{p+2}|u|^{p+2} \, dx$$

- Same subcritical vs critical divide as before.
  - $p < \frac{4}{d-2}$: LWP with lifespan $T = C(\|u_0\|_\Sigma)$
  - $p = \frac{4}{d-2}$: LWP with lifespan $T = C(u_0)$ (zoom in at $x = 0$).
  - $p < \frac{4}{d-2}$ investigated extensively by R. Carles ('02-'11).
  - $p = \frac{4}{d-2}$: GWP for radial initial data. Also scattering for $V = -\frac{1}{2}x^2$ (Killip-Visan-Zhang '09).
Call a potential \( V \) approximately quadratic if
- \( \| \partial^\alpha V \|_\infty \leq C_\alpha \) for all \( |\alpha| \geq 2 \).
- \( V(x) \geq \delta |x|^2 \) for some \( \delta > 0 \).

**Theorem (J ’14)**

For approximately quadratic \( V \), the equation

\[
 i \partial_t u = \left( -\frac{1}{2} \Delta + V \right) + |u|^{\frac{4}{d-2}} u, \quad u_0 \in \Sigma
\]

is globally wellposed. That is, for each \( u_0 \in \Sigma \) there is a unique \( u \in C(\mathbb{R}; \Sigma(\mathbb{R}^d)) \) such that (with \( A = -\frac{1}{2} \Delta + V \))

\[
 u(t) = e^{-itA} u_0 - \int_0^t e^{-i(t-s)A} |u(s)|^{\frac{4}{d-2}} u(s) \, ds
\]
Other NLS with broken symmetries

- Mass-critical gKdV (Killip-Kwon-Shao-Visan ’09)
- Cubic 2d Klein-Gordon (Killip-Stovall-Visan ’10)
- Quintic NLS (energy-critical) on various 3d domains:
  - $H^3$ (Ionescu-Pausader-Staffilani ’12)
  - $\mathbb{R}^3 \setminus$ smooth convex obstacle (Killip-Visan-Zhang ’12)
  - $T^3$ (Herr-Tataru-Tzvetkov ’10, Ionescu-Pausader ’12)
- Symmetries restored in certain limiting regimes:
  - E.g. NLS on $T^3$ (no scaling symmetry) “looks like” NLS on $\mathbb{R}^3$ at small length scales.
- Today:
  \[
i \partial_t u = (-\frac{1}{2} \Delta + V)u \text{ looks like } i \partial_t u = (-\frac{1}{2} \Delta + V(x_0))u
  \]
  if we zoom in at $x_0$.  

Jao (UCLA)
Strategy

- Seek a priori bound on

\[ S_I(u) := \int \int_{I \times \mathbb{R}^d} |u(t, x)|^{\frac{2(d+2)}{d-2}} \, dx \, dt \]

uniformly over all intervals \( I \) with \( |I| \leq 1 \). Define

\[ E_c = \sup \{ E : E(u_0) \leq E \Rightarrow \sup_{|I| \leq 1} S_I(u) \leq C(E) < \infty \} \]

Want \( E_c = \infty \). Small-data/perturbative theory \( \Rightarrow E_c > 0 \) due to Strichartz estimate for solns to linear eqn \( i \partial_t u_{lin} = (-\frac{1}{2} \Delta + V) u_{lin} \)

\[ \| u_{lin} \|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbb{R}^d)} \leq C_{|I|} \| u_{lin}(0) \|_{\Sigma}, \]

- Assume for contradiction that \( 0 < E_c < \infty \). Fix \( I = [-\frac{1}{2}, \frac{1}{2}] \), and consider \( u_n \) with \( E(u_n) \to E_c, S_I(u_n) \to \infty \).
Intuition originally due to Bourgain ('98): all the energy must asymptotically concentrate in a single “bubble.” Caricature:
\[ u_n = \lambda_n^{-\frac{d-2}{2}} v(\lambda_n^{-2} t, \lambda_n^{-1} (x - x_n)) + o(1) \] for some “profile” \( v \) with \( E(v) = E_c \).

Suppose on the contrary that \( u_n = v_n + \tilde{v}_n \), where \( v_n \) concentrates at origin and \( \tilde{v}_n \) concentrates at some \( x_0 \neq 0 \).

\[ E(u_n) = E(v_n) + E(\tilde{v}_n) + o(1) \] so \( E(v_n), E(\tilde{v}_n) \leq E_c - \delta \).

By induction on energy hypothesis and stability theory
\[ S_I(u_n) = S_I(v_n) + S_I(\tilde{v}_n) + o(1) \] bounded, contradiction.

Profile decomposition: Every seq of solns \( u_n \) with \( E(u_n) \to E_c \) and \( S_I(u_n) \to \infty \) splits into a sum of essentially independent “profiles”
\[ u_n = \sum_{j=1}^{\infty} u_n^j + o(1), \] each \( u_n^j \) with characteristic \((t_n^j, x_n^j, \lambda_n^j)\).
Profile decompositions

- Keraani ('01): Given solutions $u_n$ to $i \partial_t u = -\frac{1}{2} \Delta u + |u|^{\frac{4}{d-2}} u$ with $E(u_n)$ bdd and $S_I(u_n) \geq C > 0$, there there exist solutions $\nu^j$ and parameters $(t_n^j, x_n^j, \lambda_n^j)$ such that

$$u_n = \sum_{j=1}^{J} \lambda_n^{\frac{d-2}{2}} \nu^j \left( \frac{t-t_n^j}{(\lambda_n^j)^2}, \frac{x-x_n^j}{\lambda_n^j} \right) + o(1)$$

and $(t_n^j, x_n^j, \lambda_n^j)$, $(t_n^k, x_n^k, \lambda_n^k)$ “decouple” in the limit $n \to \infty$. E.g. $(0, 0, 0)$ and $(0, 0, n)$.

- Have analogous decomposition for $i \partial_t u = (-\frac{1}{2} \Delta + V) u + |u|^{\frac{4}{d-2}} u$, but hairier due to broken translation and scaling symmetry.

- Analogous decompositions used to treat other large-data problems (e.g. NLS outside a convex obstacle, NLS on $T^3$, NKLG).
Reducing to the constant potential eqn

- Profile decomposition and induction on energy: if \( E(u_n) \to E_c \), \( S_I(u_n) \to \infty \), then there exists \( \phi \in \Sigma \)

\[
u_n(0) = \lambda_n^{-\frac{d-2}{2}} \phi(\lambda_n^{-1}(x - x_n)) + o(1)\]

with \( E(\phi) = E_c \) and either

(i) \( \lambda_n \equiv 1, x_n \equiv 0 \), or

(ii) \( \lambda_n \to 0, |x_n|/\lambda_n \) bounded.

- In first case, let \( u_c \) be the max lifespan soln with \( u_c = \phi \).
  - Stability theory \( \Rightarrow S_I(u_c) = \lim_n S_I(u_n) = \infty \).
  - But can also show that \( \{ u_c(t) : t \in I \} \subset \Sigma \) is precompact. By stability theory, we can glue in local solns at the endpoints to extend \( u_c \), \( \Rightarrow \Leftarrow \).

- \( \lambda_n \to 0 \): potential \( V \) can be regarded as essentially constant.
Recovering the linear constant potential eqn

**Proposition**

Consider parameters $(t_n, x_n, \lambda_n)$ with $\lambda_n \to 0$, $|t_n| \lesssim \lambda_n^2$, and $|x_n| \lesssim \lambda_n^{-1}$, and define $G_n \phi = \lambda_n^{-d/2} \phi\left(\frac{x-x_n}{\lambda_n}\right)$. Then for all $\phi \in C_0^\infty$

$$\|e^{it_n(\frac{1}{2}\Delta - V)} G_n \phi - e^{-it_n V(x_n)} e^{\frac{it_n \Delta}{2}} G_n \phi\|_\Sigma \to 0 \; \forall \phi \in C_0^\infty.$$  

- Interpolate with Strichartz estimate: if $u_n^A(t) = e^{it(\frac{1}{2}\Delta - V)} G_n \phi$ and $u_n^\Delta(t) = e^{\frac{it \Delta}{2}} G_n \phi$, then
  $$S_{[-\lambda_n^2, \lambda_n^2]}(u_n^A - e^{-itV(x_n)} u_n^\Delta) \to 0.$$  

- Main ingredient: oscillatory integral representation of $e^{it(\frac{1}{2}\Delta - V)}$ for subquadratic $V$ (Fujiwara '80).
Recovering the nonlinear constant potential eqn

**Proposition**

Let \( u_n \) be solutions to

\[
i \partial_t u = \left( -\frac{1}{2} \Delta + V \right) u + |u|^{\frac{4}{d-2}} u.
\]

with \( E(u_n) \to E_c \) on some fixed cpt interval \( I = [-\delta_0, \delta_0] \), with

\[
u_n(0) = \lambda_n^{-\frac{d-2}{2}} \phi\left( \frac{x_n}{\lambda_n} \right), \quad \lambda_n \to 0, \quad \lambda_n |x_n| \lesssim 1.
\]

Then \( \limsup_n S_I(u_n) < \infty \).

Pf: By stability theory, suffices to find approximate solutions \( \tilde{u}_n \) with

\[
i \partial_t \tilde{u}_n = A\tilde{u}_n + |\tilde{u}_n|^{\frac{4}{d-2}} \tilde{u}_n + r_n
\]

with \( r_n \) small and \( S_I(\tilde{u}_n) \) bounded.
Recovering the nonlinear constant potential eqn

\[ i\partial_t u = (-\frac{1}{2}\Delta + V)u + |u|^\frac{4}{d-2}u, \]

\[ u_n(0) = \lambda_n^{-\frac{d-2}{2}} \phi(\frac{x_n}{\lambda_n}), \quad \lambda_n \to 0, \quad \lambda_n|x_n| \lesssim 1. \]

- Solution \( w \) to \( i\partial_t w = -\frac{1}{2}\Delta w + |w|^\frac{4}{d-2}w \) with \( v(0) = \phi \) is known to be global and \( S_R(v) < \infty \).

- \( \tilde{u}_n = e^{-itV(x_n)}\lambda_n^{-\frac{4}{d-2}}v(\lambda_n^{-2}t, \lambda_n^{-1}(x - x_n)) \) good approx. soln on \([-T_0\lambda_n^2, T_0\lambda_n^2]\), with \( \limsup S[-T_0\lambda_n^2, T_0\lambda_n^2](\tilde{u}_n) < \infty \).

- For \( |t| \gg T_0\lambda_n^2 \), expect soln to have dispersed. Indeed \( \tilde{u}_n = e^{-i(t-T_0\lambda_n^2)}A\tilde{u}_n(T_0\lambda_n^2) \) is a good approx. soln on \([T_0\lambda_n^2, \delta_0]\), and \( S[T_0\lambda_n^2, \delta_0](\tilde{u}_n) \) is bounded by Strichartz.

- Thus get approx soln \( \tilde{u}_n \) on \([-T_0\lambda_n^2, T_0\lambda_n^2]\) with bdd spacetime norm.