Schrödinger’s equation with random potentials

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Pasadena
February 2015
Introduction

Consider the linear Schrödinger equation in $\mathbb{R}^d$ with random time-dependent potential

$$i\partial_t \psi(x, t) - \Delta \psi(x, t) + V_\omega(x) \psi(x, t) = 0, \quad \psi(0) = \psi_0.$$  

Here $V_\omega := V(x, X_t)$. Conserved quantities for constant $V$:

$$M[\psi] := \int_{\mathbb{R}^d} |\psi(x, t)|^2 \, dx$$

$$E[\psi](t) := \int_{\mathbb{R}^d} |\nabla \psi(x, t)|^2 + V(x, t)|\psi(x, t)|^2 \, dx.$$  

Dispersive inequalities (same with more derivatives):

$$\|\psi\|_{L^\infty_t L^2_x} + \|P_c \psi\|_{L^2_t L^{2d/(d-2)}_x} \lesssim \|\psi\|_2 \text{ (Strichartz)}$$

$$\|D^{1/2} P_c \psi\|_{L^2_t L^2_x(Q)} \lesssim |Q|^{1/2d} \|\psi\|_2 \text{ (local smoothing)}.$$  

Constant $V$: $E$ and $M$ conserved; $P_c \psi$ disperses.

Time-dependent $V$: $M$ is conserved.
Main result

Consider the equation on $\mathbb{R}^3$ (or $d \geq 3$)

$$i\psi_t - \Delta \psi + V(x, X_t)\psi + \epsilon(\chi * |\psi|^2)\psi = 0, \ \psi(0) = \psi_0 \text{ given.}$$

$X_t$ standard Brownian motion on bounded subset of Riemannian manifold; nontrivial $V(x, y) \in C_y(L_x^\infty \cap L_x^1)$; Hartree-type potential with small coupling constant.

For any $\psi_0 \in L^2$ and $|\epsilon| < \epsilon(\|\psi_0\|_2)$, there is a.s. a global solution $\psi$ s.t.

$$\mathbb{E}\|\psi\|_{L_t^2 L_x^{6,2}}^2 \lesssim \|\psi_0\|_2^2.$$

Moreover, if $A_y V(x, y) \in L_y^\infty (L_x^1 \cap L_x^\infty)$, then the energy remains bounded on average:

$$\mathbb{E}(\|\nabla \psi_\omega(t)\|_{L_x^2}^2) \leq \|\nabla \psi_0\|_{L_x^2}^2 + C\|\psi_0\|_{L_x^2}^2.$$

Model problem: $-\Delta + V_0(x) + g(W_t)V_1(x)$, $V_0 \in C_0^\infty$ large fixed potential, $W_t$ standard Brownian motion on $\mathbb{T}^1$, $V_1$ perturbation, coupling $g(y) = 0.1 \sin(y)$.

Other simple perturbations: e.g. $-\Delta + V_0(x) + V_1(x - W_t)$, rotations, dilations. Can treat the general case stated above.
Known results

1974 Ovchinnikov.
1985, 1986 Pillet: started the study, Feynman-Kac formula, linear $L^2$ wave operators.
1989, 1990 Cheremshantsev: Brownian motion over the whole space.
2010 De Roeck, Fröhlich, Pazzo: For a quantum particle interacting with infinitely many thermal reservoirs, they proved a central limit theorem, diffusive scaling for the second momentum, with no corrections, and a distribution law for finite energy states.
2013 Beceanu, Soffer: Strichartz estimates, other properties; Brownian motion on the whole space.
Proof outline

Consider the inhomogenous linear equation with random potential $V_\omega := V(x, X_t)$

$$i\partial_t \psi_\omega - \Delta \psi_\omega + V_\omega \psi_\omega = \Psi_\omega, \, \psi_\omega(0) := \psi_0.$$

Define

$$g(x, y, t) := \mathbb{E}(\psi(x, t) | X_t = y),$$

$$f(x_1, x_2, y, t) := \mathbb{E}(\psi(x_1, t)\overline{\psi}(x_2, t) | X_t = y).$$

Conditional expectations of $\psi$, respectively of the density matrix $\psi \otimes \overline{\psi}$, at time $t$ and under the condition that $X_t = y$. 
As shown by Pillet,

\[ i\partial_t g - \Delta_x g - iA_y g + V(x, y)g = G, \quad g(0) := \psi_0(x)\mu_0(y), \]
\[ i\partial_t f - \Delta_{x_1} f + \Delta_{x_2} f - iA_y f + (V \otimes 1 - 1 \otimes V)f = F, \]
\[ f(0) := \psi_0(x_1)\overline{\psi}_0(x_2)\mu_0(y), \]

where \( \mu_0(y) \) is the initial distribution of \( X_t \), i.e. of \( X_0 \), and

\[ G(x, y, t) := \mathbb{E}(\psi(x, t) \mid X_t = y), \]
\[ F(x_1, x_2, y, t) := \mathbb{E}(\psi(x_1, t)\overline{\psi}(x_2, t) - \psi(x_1, t)\overline{\psi}(x_2, t) \mid X_t = y). \]
Pillet also proved the following Feinman-Kac-type formula:

\[ \mathbb{E} \int |\psi_\omega(x, t)|^2 |V_\omega(x, t)| \, dx \, dt = \int f(x, x, y, t) |V(x, y)| \, dx \, dy \, dt. \]

We compute the right-hand side by using the equation of \( f \). The left-hand side controls Strichartz estimates:

\[ \mathbb{E} \|\psi_\omega\|^2_{L_t^2 L_x^{6,2}} \lesssim \|\psi_0\|^2_2 + \mathbb{E} \int |\psi_\omega(x, t)|^2 |V_\omega(x, t)| \, dx \, dt. \]
Consider the solution $g$ to

\[ i\partial_t g - \Delta_x g + iA_y g + V(x_1, y)g = 0. \]

Enough to prove that $g \in L^2_{\omega} L^2_{t,y} L^6_{x,2}$. 

Non-triviality assumption on $V$: For almost every $x$ in some open set $\mathcal{O}$ there exist $y_1, y_2$ such that $V(x, y_1) \neq V(x, y_2)$. Then $-\Delta_x + iA_y + V$ has no bound states for $\Im \lambda \leq 0$.

Proof: bound state $\phi(x, y) \implies$ independent of $y \implies$ for each $y$ solves $(-\Delta + V(x, y))\phi(x) = \lambda \phi(x)$. $V(x, y)$ are distinct $\implies \phi = 0$ on open domain $\implies \phi \equiv 0$. 

M. Beceanu J. Fröhlich A. Soffer Random Potentials 8/15
Theorem
Let $\mathcal{W}_X := \langle t \rangle^{-3/2} L_t^\infty B(X)$. Suppose $T \in \mathcal{W}_X$ is s.t.
$$\lim_{\delta \to 0} \| T(\rho) - T(\rho - \delta) \|_{\mathcal{W}_X} = 0.$$  
If $I + \hat{T}(\lambda)$ is invertible in $B(X)$ for every $\lambda \in \mathbb{R}$, then $1 + T$ possesses an inverse in $\overline{\mathcal{W}_X}$ of the form $1 + S$.

Lemma
Consider the equation, for $V \in C_y(L_x^\infty \cap L_x^1)$ and nontrivial,
$$i \partial_t f - \Delta_x f + i A_y f + V(x, y) f = 0, \quad f(0) = f_0 \in L_y^1 L_x^2.$$

Then Strichartz:
$$\| f \|_{L_y^2 L_t^2 L_x^{6.2}} \lesssim \| f_0 \|_{L_y^2 L_x^2}.$$

Moreover,
$$\| f \|_{L_y^p(L_x^2 + L_x^\infty)} \lesssim \langle t \rangle^{-3/2} \| f_0 \|_{L_y^p(L_x^1 \cap L_x^2)}.$$
Rewrite Duhamel’s identity symmetrically

\[ I - \chi_{t>0} i |V|^{1/2}(x, y) e^{it(-\Delta_x + V(x,y) + iA_y)} |V|^{1/2} \text{sgn } V(x, y) = \]

\[ = (I + \chi_{t>0} i |V|^{1/2}(x, y) e^{it(-\Delta_x + iA_y)} |V|^{1/2} \text{sgn } V(x, y))^{-1}. \]

Apply Wiener’s theorem. Leads to

\[ I - \chi_{t>0} i |V|^{1/2}(x, y) e^{it(-\Delta_x + iA_y + V(x,y))} |V|^{1/2} \text{sgn } V(x, y) \]

being \(L^1_{t,y} L^2_x\)-bounded. Conclusion follows by Duhamel.
Repeat for density matrix. Let $V \otimes 1 + 1 \otimes V = V_1 V_2$, where

$$V_2 := \begin{pmatrix} |V|^{1/2} \text{sgn } V \otimes 1 \\ 1 \otimes |V|^{1/2} \text{sgn } V \end{pmatrix}, \quad V_1 := \begin{pmatrix} |V|^{1/2} \otimes 1 & 1 \otimes |V|^{1/2} \end{pmatrix}.$$ 

The free resolvent is

$$R^A(\lambda) := (-\Delta_{x_1} + \Delta_{x_2} - iA_y - \lambda)^{-1}.$$ 

Need to prove that $V_2 R^A(\lambda) V_1$ is $L^2$-bounded and, after taking the Fourier transform,

$$I + V_2 R^A(\lambda) V_1$$

is invertible for every $\lambda$. Matrix form

$$I + T = \begin{pmatrix} I + T_{11} & T_{12} \\ T_{21} & I + T_{22} \end{pmatrix}.$$
\[ I + i\hat{T} = \begin{pmatrix} I + i\hat{T}_{11} & 0 \\ 0 & I + i\hat{T}_{22} \end{pmatrix} \begin{pmatrix} I \quad i(I + i\hat{T}_{11})^{-1}\hat{T}_{12} \\ i(I + i\hat{T}_{22})^{-1}\hat{T}_{21} \quad I \end{pmatrix}. \]

Diagonal terms are invertible by reduction to previous case:

\[ (I + iT_{11})^{-1} = I - i\chi_{t>0}e^{it\Delta x_2}|V|^{1/2}(x_1)e^{it(-\Delta x_1 + V(x_1,y) + iA_y)}|V|^{1/2}\text{sgn } V(x_1). \]

Now we need to study

\[ I + \begin{pmatrix} 0 \\ i(I + i\hat{T}_{22})^{-1}\hat{T}_{21} \end{pmatrix} =: I + \begin{pmatrix} 0 & i\hat{S}_1 \\ i\hat{S}_2 & 0 \end{pmatrix} =: I + i\hat{S}. \]

Then

\[ S^2 = \begin{pmatrix} S_1S_2 & 0 \\ 0 & S_2S_1 \end{pmatrix}. \]

After squaring, these off-diagonal terms are compact. One can use Fredholm’s alternative for \( I + S^2 = (I - iS)(I + iS) \) (same for both). Just as importantly, Wiener’s theorem applies to \( I + S^2 \).
Write \((I + iT)^{-1}\) in terms of \((I + S^2)^{-1}\) and the diagonal terms \((I + iT_{kk})^{-1}\). Get almost all integrable components, convolved with something explicit. Use this to evaluate \(f\).

In fact much more complicated.

Conclusion:

\[
\int f(x, x, y, t)|V(x, y)| \, dx \, dy \, dt \lesssim \|\psi_0\|_{L^2}^2.
\]
Well-posedness for Hartree with small initial data

Proof: Contraction scheme. By $L^2$ norm conservation, a priori $\psi \in L^\infty_t L^2_x$. The inhomogenous source term is small in $L^1_\omega L^1_t J_{1x_1,x_2}$ because

$$\|(\chi \ast |\psi|^2)\psi \otimes \psi\|_{L^1_\omega L^1_t J_{1x_1,x_2}} \lesssim \|\psi\|^2_{L^2_\omega L^2_t L^{6,2}_x} \|\psi\|^2_{L^\infty_\omega L^\infty_t L^2_x}.$$ 

and $\psi$ belongs to $L^2_\omega L^2_t L^{6,2}_x$ by Strichartz.
Thank you for your attention!