The generic action of a free group is hyperfinite

(joint with Sumun Iyer)

Forte Shinko

UCLA

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The generic closed subspace of $[0,1]^{\mathbb{N}}$ is homeomorphic to $2^{\mathbb{N}}$.

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Topology on space of actions = subspace topology from $\mathrm{Homeo}(2^{\mathbb{N}})^\Gamma$.

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Example

$F_2 \curvearrowright 2^{F_2}$ non-hyperfinite (F_2 nonamenable).

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Measure-hyperfinite is equivalent to hyperfinite.

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Theorem (Frisch-Kechris-Shinko-Vidnyánszky)

The generic subshift of $(\text{Hilbert cube})^{F_n}$ *is measure-hyperfinite.*

Proof: Generic subshift is measure-hyperfinite

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$$\{\text{measure-hyperfinite}\}$$
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Theorem (Kechris-Louveau-Woodin)

$\mathbf{\Sigma}^1_1 \implies G_\delta$ *for a σ-ideal of compact sets.*

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There is a comeager conjugacy class of actions $F_n \curvearrowright 2^{\mathbb{N}}$.

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Uses framework of projective Fraïssé theory.

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Surjective structure = finite set equipped with two surjective relations.

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Non-amalgamable structure:

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Sufficient condition for amalgamation: **No forks**.

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then the generic $F_2 \curvearrowright 2^{\mathbb{N}}$ factors onto X.

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Theorem (IS)

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Pull back hyperfiniteness from ∂F_2.

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Doesn't work for F_∞.

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Hence it has Borel asymptotic dimension 1.
(also finite rank F_n)

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and we get hyperfiniteness.

Thank you!