Measurable Vizing's theorem

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Classical edge colorings

Theorem (Vizing, Gupta)

Let $G = (V, E)$ *be a (simple) graph of degree bounded by* $\Delta < +\infty$*. Then there is a map* $c : E \to [\Delta + 1]$ *such that* $c(e) \neq c(f)$ *whenever* $e \cap f \neq \emptyset$*.*[1]

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(a) such a map c is called a *proper edge coloring*,

(b) chromatic index of G, $\chi'(G)$, is the smallest number of colors needed for a proper edge coloring of G,

(c) Vizing's theorem $\Rightarrow \chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$, where $\Delta(G) = \max\{\deg_G(v) : v \in V\}$

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Measurable edge colorings

A *Borel graph* $\mathcal{G}$ is a triplet $(V, \mathcal{B}, E)$, where $(V, \mathcal{B})$ is a standard Borel space, (V, E) is a graph and E is a Borel subset of $[V]^2$ (the set of unordered pairs of V endowed with the Borel structure inherited from $V \times V$).

A *proper Borel edge coloring of* $\mathcal{G}$ *with* k *colors* is a Borel map $c : E \to [k]$ that is a proper edge coloring.

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A *proper Borel edge coloring of* $\mathcal{G}$ *with* k *colors* is a Borel map $c : E \to [k]$ that is a proper edge coloring.

- the **Borel chromatic index of** $\mathcal{G}$, $\chi'_{\mathcal{B}}(\mathcal{G})$, is the smallest number of colors needed for a proper Borel edge coloring of $\mathcal{G}$,

Borel colorings

Theorem (Kechris–Solecki–Todorčević)

Let $\mathcal{G}$ be a Borel graph such that $\Delta(\mathcal{G}) < +\infty$. Then $\chi'_{\mathcal{B}}(\mathcal{G}) \leq 2\Delta(\mathcal{G}) - 1$.

- Laczkovich gave an example of 2-regular acylic Borel bipartite Borel graph $\mathcal{G}$ such that $\chi'_{\mathcal{B}}(\mathcal{G}) = 3$.

Theorem (Marks)

For every $\Delta > 2$ and every $k \in \{\Delta, ..., 2\Delta - 1\}$, there is a Δ-regular acyclic Borel bipartite Borel graph $\mathcal{G}$ such that $\chi'_{\mathcal{B}}(\mathcal{G}) = k$.

- In particular, Vizing's theorem **fails** in the Borel context.

Measurable analogues of Vizing's theorem

Question (Abért)

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Question (Kechris–Marks, Problem 6.13)

Let $\mathcal{G}$ be a Borel graph on a Polish space with $\Delta(\mathcal{G}) < +\infty$. Is it true that $\chi'_M(\mathcal{G}) \leq \Delta + 1$? Is it true that $\chi'_{BM}(\mathcal{G}) \leq \Delta + 1$?

Measurable setting

Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph and μ be a *Borel probability measure* on $(V, \mathcal{B})$.

Definition

The μ-**measurable chromatic index of** $\mathcal{G}$, $\chi'_\mu(\mathcal{G})$ is defined as the minimum $k \in \mathbb{N}$ such that there is a μ-null set $X \subseteq V$ such that $\chi'_\mathcal{B}(\mathcal{G} \restriction (V \setminus X)) = k$.

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- the *measurable chromatic index of* $\mathcal{G}$, $\chi'_M(\mathcal{G})$, is defined as supremum of $\chi'_\mu(\mathcal{G})$ over all Borel probability measures μ on $(V, \mathcal{B})$,
- $\Delta(\mathcal{G}) \leq \chi'(\mathcal{G}) \leq \chi'_\mu(\mathcal{G}) \leq \chi'_M(\mathcal{G}) \leq \chi'_\mathcal{B}(\mathcal{G}) \leq 2\Delta(\mathcal{G}) - 1$

Graphings

Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph and μ be a *Borel probability measure* on $(V, \mathcal{B})$.

- We say that μ is $\mathcal{G}$-invariant if

$$\int_A \deg_B(v)\ d\mu(v) = \int_B \deg_A(w)\ d\mu(w)$$

holds for every two Borel sets $A, B \subseteq V$, where $\deg_Y(v) = |\{w \in Y : \{v, w\} \in E\}|$.

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In this case, the quadruple $(V, \mathcal{B}, E, \mu)$ is called a **graphing**.

Positive results for graphings

Theorem (Csóka–Lippner–Pikhurko)

Let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ *be a graphing. Then*

$$\chi'_\mu(\mathcal{G}) \le \Delta(\mathcal{G}) + O(\sqrt{\Delta(\mathcal{G})}),$$

and $\chi'_\mu(\mathcal{G}) \le \Delta(\mathcal{G}) + 1$ *if* $\mathcal{G}$ *does not contain odd cycles.*

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Theorem (G–Pikhurko)
Let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ be a graphing. Then $\chi'_\mu(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1$.

This answers the question of Abért.

A sample of related positive results

Many new results in recent years (work of Bencs, Bernshteyn, Bowen, Chandgotia, Gao, Hrušková, Jackson, Krohne, Qian, Rozhoň, Seward, Thornton, Tóth, Unger, Weilacher, ...).

For example:

- Bernshteyn extended and applied the method of G–Pikhurko in the context of distributed algorithms.
- Free Borel actions of $\mathbb{Z}^d$ admit proper **Borel** edge coloring with $2d$ colors (independently by Bencs–Hrušková–Tóth, and Chandgotia–Unger, and G–Rozhoň, and Weilacher).
- Qian and Weilacher found connections of the topological relaxation to computable combinatorics which allowed them to derive an upper bound of $\Delta(\mathcal{G}) + 2$ colors for the Baire measurable analogue of Vizing's theorem.

Main result

Theorem
Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < +\infty$ and μ be a Borel probability measure on $(V, \mathcal{B})$. Then $\chi'_\mu(\mathcal{G}) \leq \Delta + 1$.

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Ingredients in the proof:

(I) technique of augmenting *iterated* Vizing chains introduced in G–Pikhurko,

(II) replacing μ with an equivalent but “tame” measure ν.

General strategy

A *partial* Borel edge coloring (of $\mathcal{G}$) is a Borel map $c : \mathrm{dom}(c) \to [\Delta(\mathcal{G}) + 1]$ that satisfies $c(e) \neq c(f)$ whenever $e \cap f \neq \emptyset$ and $e, f \in \mathrm{dom}(c)$, where $\mathrm{dom}(c)$ is a Borel subset of E.

If we do not want to specify the domain we write simply $c; E \to [\Delta(\mathcal{G}) + 1]$, and we set $U_c = V \setminus \mathrm{dom}(c)$.

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Strategy: Inductively improve given partial Borel coloring.

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(a) Assign to each $e \in U_c$ an **augmenting** connected subgraph (a chain of edges) $V_c(e) \subseteq \mathrm{dom}(c)$ with the property:

- (•) $\exists\ c_e; E \to [\Delta(\mathcal{G})+1]$ such that $\mathrm{dom}(c_e) = \mathrm{dom}(c) \cup \{e\}$ and $c \restriction E \setminus V_c(e) = c_e \restriction E \setminus V_c(e)$.

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(b) Take a maximal collection $\{V_c(e)\}_{e \in I}$ such that $V_c(e) \cap V_c(f) = \emptyset$ for every $e \neq f \in I$, and augment all $\{V_c(e)\}_{e \in I}$ simultaneously to create $c'; E \to [\Delta(\mathcal{G})+1]$ such that $\mathrm{dom}(c') = \mathrm{dom}(c) \cup \{e\}_{e \in I}$.

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(c) Start with $c_0 = \emptyset$, and iterate this procedure to create a sequence of partial colorings $\{c_n\}_{n=1}^{\infty}$ with the hope that

$$c(e) = \lim_{n \to \infty} c_n(e)$$

is defined off of a μ-null set.

(I) Augmenting chains

Let $e = \{x, y\} \in U_c$, and pick

- $\alpha \in m_c(x) = [\Delta(\mathcal{G}) + 1] \setminus \{c(f) : x \in f\}$ (colors missing at x),
- $\beta \in m_c(y) = [\Delta(\mathcal{G}) + 1] \setminus \{c(f) : y \in f\}$.

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Define $P_c(x, e)$ to be the concatenation of e and the maximal α/β path starting at y.

(I) Augmenting chains

If we are lucky and $P_c(x, e)$ does not come back to x, then $P_c(x, e)$ is **augmenting**.

(I) Augmenting chains

Up to a little reshuffling of colors around x, this can be always achieved. The augmenting chain is called *Vizing's chain*, and it is of the form $W_c(x, e) = F^\frown P(\alpha/\beta)$. (Augmenting these chains one-by-one gives the proof of Vizing's theorem for finite graphs.)

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How do augmenting chains connect with the measure μ?

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Proposition

Let $\mathcal{G}$ be a graphing and $c; E \to [\Delta(\mathcal{G})+1]$ be such that $|W_c(x,e)| \geq L + \Delta$ for some $L \in \mathbb{N}$. Then $\mu(U_c) \leq \frac{2\Delta^3}{L}$.

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Proof Define an auxiliary graph $\mathcal{H}_c$ with vertex set $U_c \sqcup \mathrm{dom}(c)$ and $(e,f) \in \mathcal{H}_c$ if $f \in P$ where $W_c(x,e) = F^\frown P$.

(I) Approximate edge colorings

We have $\deg_{\mathcal{H}_c}(e) \geq L$ for every $e \in U_c$ and $\deg_{\mathcal{H}_c}(f) \leq 2\Delta^3$ for every $e \in \mathrm{dom}(c)$.

The fact that μ is $\mathcal{G}$-invariant gives

$$L\mu(U_c) \leq \int_{U_c} \deg_{\mathcal{H}_c}(e) = \int_{\mathrm{dom}(c)} \deg_{\mathcal{H}_c}(f) \leq 2\Delta^3.$$

(I) Approximate edge colorings

Elek–Lippner type argument shows that given $d; E \to [\Delta(\mathcal{G})+1]$, it is always possible to modify colors of at most $O(L)\mu(U_d)$ edges to produce $c; E \to [\Delta(\mathcal{G})+1]$ such that $|W_c(x,e)| \geq L$ for every $e \in U_c$ and $\mathrm{dom}(d) \subseteq \mathrm{dom}(c)$.

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Proposition (Approximate Vizing for graphings)

For every $\epsilon > 0$, there is a partial Borel proper edege coloring $c; E \to [\Delta + 1]$ such that $\mu(U_c) \leq \epsilon$.

(I) Iterated Vizing chains

Unfortunately, the price that we have to pay for the modification, $O(L)$, is of the same order as $\mu(U_c)^{-1}$. Hence iterating this process will not produce $d(e) = \lim_{n\to\infty} d_n(e)$.

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New idea in G–Pikhurko yields $\deg_{\mathcal{H}_c}(e) \geq L^2$ for $e \in U_c$:

(II) Quasi-invariant measures

Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < +\infty$. Then the connectivity component relation of $\mathcal{G}$, $F_{\mathcal{G}}$, is a **countable Borel equivalence relation (CBER)**, i.e., the connectivity component $[v]_{\mathcal{G}}$ of each $v \in V$ is at most countable.

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Definition

Let μ be a Borel probability measure on $(V, \mathcal{B})$. We say that μ is *$\mathcal{G}$-quasi invariant* if $\mu([A]_{\mathcal{G}}) = 0$, whenever $\mu(A) = 0$.

(II) Quasi-invariant measures

Fundamental tool $\rightsquigarrow$ Radon-Nikodym cocycle

A Borel function $\rho_\mu : F_{\mathcal{G}} \to \mathbb{R}_{>0}$ with the property that

$$\mu(g(C)) = \int_C \rho_\mu(x, g(x))\ d\mu(x)$$

for every $C \in \mathcal{B}$ and injective Borel map $g : C \to V$ such that $(v, g(v)) \in F_{\mathcal{G}}$.

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$\rightsquigarrow$ chains of large weight can be very short (the main issue for iterated Vizing chains)

(II) Bounded cocycle

Theorem

Let $\mathcal{G}$ be a Borel graph and μ be a $\mathcal{G}$-quasi-invariant Borel probability measure on $(V, \mathcal{B})$. Then there is an equivalent Borel probability measure ν on $(V, \mathcal{B})$ such that

$$\frac{1}{4\Delta} \leq \rho_\nu(x, y) \leq 4\Delta$$

for every edge $(x, y) \in E$.

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$\rightsquigarrow$ if $\mathrm{dist}_{\mathcal{G}}(x, y) = k \in \mathbb{N}$, then $\rho_\nu(x, y) \leq (4\Delta)^k$, where $\mathrm{dist}_{\mathcal{G}}$ is the graph distance on $\mathcal{G}$.

$\rightsquigarrow$ chains of weight L have size $\Omega(\log(L))$

(II) Bounded cocycle

Sketch of the argument:

(a) Averaging the cocycle ρ_μ to define everywhere positive $\Omega \in L^1(\mu)$ such that $\frac{1}{\int \Omega \; d\mu}\Omega = \frac{\nu}{\mu}$.

(b) Showing that $\rho_\nu(x, y) = \frac{\Omega(y)}{\Omega(x)}\rho_\mu(x, y)$ has the desired properties. Technical but direct computation once we describe Ω.

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How to get the uniform distribution by averaging?

Define

$$\Omega(k) = \sum_{j=1}^{n} \rho_\alpha(k,j) = \frac{1}{\alpha(k)}.$$

Then we have $\int_n \Omega d\alpha = n$ and β that satisfies $\frac{1}{n}\Omega = \frac{d\beta}{d\alpha}$ is the uniform distribution on n.

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Indeed, we have

$$\rho_\beta(k,\ell) = \frac{\Omega(\ell)}{\Omega(k)}\rho_\alpha(k,\ell) = \frac{\alpha(k)}{\alpha(\ell)}\frac{\alpha(\ell)}{\alpha(k)} = 1.$$

(II) Bounded cocycle

(a) For $v \in V$, we define

$$\Omega(v) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\mathrm{dist}(v,w)=k} \frac{1}{\Delta^k} \rho_\mu(v, w).$$

Need to show that $\Omega \in L^1(\mu)$, in particular, $\Omega(v)$ is finite μ-almost everywhere.

In reality formula is more complicated that is why we get the estimate 2Δ on the next slide.

(II) Bounded cocycle

(b) We have for every edge $(x, y) \in E$

$$\begin{aligned}
\Omega(y)\rho_\mu(x,y) &= \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\mathrm{dist}(y,w)=k} \frac{1}{\Delta^k} \rho_\mu(y,w)\right) \rho_\mu(x,y) \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\mathrm{dist}(y,w)=k} \frac{1}{\Delta^k} \rho_\mu(x,w) \\
&\leq 2\Delta \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\mathrm{dist}(x,w)=k} \frac{1}{\Delta^k} \rho_\mu(x,w)\right) = 2\Delta\Omega(x).
\end{aligned}$$

$\rightsquigarrow \rho_\nu(x,y) = \frac{\Omega(y)}{\Omega(x)} \rho_\mu(x,y) \leq 2\Delta$

Thank you!