Cardinalities Below the Power Set of the First Uncountable Cardinals

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Cardinalities

Definition (Cantor)
Define a proper class equivalence relation on all sets by $X \approx Y$ if and only if there is a bijection between X and Y. A cardinality is a $\approx$-equivalence class. If X is a set, then $|X| = [X]_{\approx}$ be the $\approx$-equivalence class of X.

Define $|X| \leq |Y|$ if and only if there exists an injection $\Phi : X \to Y$. $|X| < |Y|$ if and only if $(|X| \leq |Y|)$ and $\neg(|Y| \leq |X|)$.

A cardinal is an ordinal which does not inject into any smaller ordinal.

If the axiom of choice holds, then all sets can be wellordered. Thus every proper class cardinality $\mathcal{X}$ has a canonical member: the unique cardinal $\kappa \in \mathcal{X}$. All cardinalities are wellordered by the injection relation.

The axiom of choice tends to erase defining characteristics of sets or make them irrelevant in distinguishing sets by size. The axiom of choice is not the only setting for study mathematical size. Other rebust framework exist which are motivated by combinatorics and descriptive set theory.

We will be mostly concerned with cardinalities of very familiar sets. These tend to be surjective images of $\mathbb{R}$ or equivalently quotients of equivalence relations on $\mathbb{R}$. Results from descriptive set theory characterizes the injections between certain quotients which have simply definable liftings.

Fact (ZF; Silver's dichotomy)
If E is a $\mathbf{\Pi}^1_1$ equivalence relation on $\mathbb{R}$, then exactly one of the following occurs.

- *$\mathbb{R}/E$ is countable.*
- *$= \leq_B E$, there is a Borel reduction $\Phi : \mathbb{R} \to \mathbb{R}$ so that $x = y$ if and only if $\Phi(x)\ E\ \Phi(y)$. (So Φ induces an injection of $\mathbb{R}$ into $\mathbb{R}/E$.)*

Define E_0 on ${}^\omega 2$ by $x\ E_0\ y$ if and only if there exists $m \in \omega$ so that for all $n \geq m$, $x(n) = y(n)$.

Fact (ZF; E_0-dichotomy; Harrington-Kechris-Louveau)
If E is a $\mathbf{\Delta}^1_1$ equivalence relation, then exactly one of the following occurs.

- *$E \leq_B =$. The reduction induces an injection of $\mathbb{R}/E$ into $\mathbb{R}$ or $\mathscr{P}(\omega)$.*
- *$E_0 \leq_B E$. The reduction induces an injection of $\mathbb{R}/E_0$ injects into $\mathbb{R}/E$.*

Determinacy

Determinacy provides a robust choiceless framework which extends the theory of “Borel cardinality” of quotients of equivalence relations to a theory for genuine cardinalities.

The following is the most basic form of the axiom of determinacy, AD.

Let $A \subseteq {}^{\omega}\omega$.

$$
\begin{array}{ccccccccc}
 & 1 & a_0 & & a_2 & & a_4 & & \cdots & \\
G_A & & & & & & & & & \vec{a} \\
 & 2 & & a_1 & & a_3 & & a_5 & \cdots &
\end{array}
$$

Player 1 wins G_A if and only if $\vec{a} \in A$. The axiom of determinacy, AD, is the assertion that for all $A \subseteq {}^{\omega}\omega$, one of the two players has a winning strategy in G_A.

Classification Program

Determinacy assumptions influences the cardinalities and combinatorics of sets which are surjective images of $\mathbb{R}$.

- Completely classify the cardinalities and their injection relation below familiar sets which are surjective images of $\mathbb{R}$.
- For familiar sets, determines its global position or relation to all other sets which are surjective images of $\mathbb{R}$.

Toward the ambitious goal of complete classification, the following problems should address.

- Distinguish the cardinalities among certain classes of familiar sets.
- Gain a sufficient understanding of the cardinality of familiar sets to investigate its regularity or cofinality.

Regularity and Cofinality

In the familiar ZFC context, the cofinality of a cardinal κ is the smallest cardinal δ so that there is a partition $\Phi : \kappa \to \delta$ so that for all $\alpha < \delta$, $|\Phi^{-1}[\{\alpha\}]| < |\kappa|$.

Definition
A set X has Y-regular cardinality if and only if for all $\Phi : X \to Y$, there exists a $y \in Y$ so that $|\Phi^{-1}[\{y\}]| = |X|$.

X has locally regular cardinality if and only if for all Y with $|Y| < |X|$, X is Y-regular.

X has globally regular cardinality if and only if for all Y with $\neg(|X| \leq |Y|)$, X is Y-regular.

Definition
The local cofinality of X is

$$\mathrm{lcof}(X) = \{Y : (\exists Z)(Z \subseteq X \wedge Z \approx Y \wedge X \text{ has } Y\text{-regular cardinality}\}.$$

If X is a set, let $\mathrm{Surj}(X)$ be the set of Y so that Y is a surjective image of X. The global cofinality of X is

$$\mathrm{gcof}(X) = \{Y \in \mathrm{Surj}(X) : X \text{ has } Y\text{-regular cardinality}\}.$$

Regularity and Cofinality

Fact (ZF)
If κ is a regular cardinal, then κ is globally regular.
$\mathrm{lcof}(\kappa) = \mathrm{gcof}(\kappa) = |\kappa| = \{X : X \approx \kappa\}$.

The search for the cofinality of a set X entails finding all Y so that X is Y-regular in which case, every set Z with $|Z| \leq |X|$ has been excluded from $\mathrm{gcof}(X)$.

If X is locally regular, then $\mathrm{lcof}(X) = |X|$. If X is globally regular, then $\mathrm{gcof}(X) = \{Y \in \mathrm{Surj}(X) : |X| \leq |Y|\}$.

Cardinality of the Power Set of ω

Fact

Assume $\mathsf{AC}_\omega^\mathbb{R}$ *and all sets of reals have the perfect set property. If* $X \subseteq \mathscr{P}(\omega)$ *is uncountable, then* $|X| = |\mathbb{R}| = |\mathscr{P}(\omega)|$*.* $\mathbb{R}$ *has* ω*-regular cardinality and hence* $\mathbb{R}$ *has locally regular cardinality.* $\mathrm{lcof}(\mathbb{R}) = |\mathbb{R}|$*.*

Assuming all sets of reals have the Baire property, well ordered union of meager sets are meagers.

Fact

Assume $\mathsf{AC}_\omega^\mathbb{R}$ *and all sets of reals have the perfect set property and the property of Baire.* $\mathbb{R}$ *is not wellorderable. Thus* $\neg(|\mathbb{R}| \leq |\omega_1|)$ *and* $\neg(|\omega_1| \leq |\mathbb{R}|)$*.* $\mathbb{R}$ *has* ON*-regular cardinality.*

Woodin generalized Harrington's proof of the Silver's dichotomy using Vopěnka forcing to show that $|\mathbb{R}|$ has a very special global relationship to all other cardinalities which are surjective images of $\mathbb{R}$.

Fact (Woodin; AD$^+$)
If X is a surjective image of $\mathbb{R}$, then exactly one of the following holds.

- *X is well orderable.*
- $|\mathbb{R}| \leq |X|$.

Theorem (AD$^+$; Chan-Jackson-Trang)
$\mathbb{R}$ has globally regular cardinality. $\mathrm{gcof}(\mathbb{R}) = \{X \in \mathrm{surj}(X) : |\mathbb{R}| \leq |X|\} = \{X \in \mathrm{surj}(\mathbb{R}) : X \text{ is not wellorderable}\}$.

Cardinality of $\mathbb{R}/E_0$

Hjorth generalized the E_0-dichotomy of H-K-L using Vopěnka forcing to establish the global position of $|\mathbb{R}/E_0|$ among all other cardinalities which are surjecive images of $\mathbb{R}$.

Fact (Hjorth; AD^+**)**
If X is a surjective image of $\mathbb{R}$, then exactly one of the following holds.

- *There exists a $\delta \in \mathrm{ON}$, $|X| \leq |\mathscr{P}(\delta)|$ (X is linearly orderable) (Tame)*
- $|\mathbb{R}/E_0| \leq |X|$. *(X is not linearly orderable) (Untame)*

The cardinality structure of $\mathbb{R}/E_0$ is completely classified by the following picture.

$$\begin{array}{c} \mathbb{R}/E_0 \\ \uparrow \\ \mathbb{R} \end{array}$$

Cardinality of $\mathbb{R}/E_0$

Fact (C-J-T)
Assume all sets of reals have the Baire property. Let Y be a linearly orderable set. Then $\mathbb{R}/E_0$ is Y-regular.

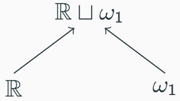
Fact (Hjorth; AD^+)
If X is a surjective image of $\mathbb{R}$, then exactly one of the following holds.

- *There exists a $\delta \in \mathrm{ON}$, $|X| \leq |\mathscr{P}(\delta)|$ (X is linearly orderable) (Tame)*
- $|\mathbb{R}/E_0| \leq |X|$. *($X$ is not linearly orderable) (Untame)*

Theorem (C-J-T; AD^+)
$\mathbb{R}/E_0$ has globally regular cardinality. $\mathrm{gcof}(\mathbb{R}/E_0) = \{X \in \mathrm{Surj}(\mathbb{R}) : |\mathbb{R}/E_0| \leq X\} = \{X \in \mathrm{Surj}(\mathbb{R}) : X$ *is not linearly orderable*$\}$.

Cardinality of $\mathbb{R} \sqcup \omega_1$

Fact (AD**)**
Assume $\mathrm{AC}_\omega^{\mathbb{R}}$ *and every sets of reals have the perfect set property and the property of Baire. The structure of the cardinality of* $\mathbb{R} \sqcup \omega_1$ *is given by the following picture.*



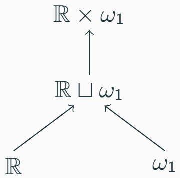
Fact (C-J-T)
Assume $\mathrm{AC}_\omega^{\mathbb{R}}$ *and every set of reals have the perfect set property and the property of Baire.* $\mathbb{R} \sqcup \omega_1$ *does not have* 2*-regular cardinality.*
$\mathrm{gcof}(\mathbb{R} \sqcup \omega_1) = \{X \in \mathrm{Surj}(\mathbb{R}) : |X| > 1\}$.

Let $\Phi : \mathbb{R} \sqcup \omega_1 \to 2$ be

$$\Phi(x) = \begin{cases} 0 & x \in \mathbb{R} \\ 1 & x \in \omega_1 \end{cases}$$

Cardinality of $\mathbb{R} \times \omega_1$

Assuming $\mathsf{AC}_\omega^\mathbb{R}$ and all sets of reals have the perfect set property and the property of Baire. The four natural subsets of $\mathbb{R} \times \omega_1$ have the following cardinality relation.



Assuming AD + Uniformization (also called $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$; Kechris), the above picture is the complete local classification of cardinality below $|\mathbb{R} \times \omega_1|$.

Main idea: Let $X \subseteq \mathbb{R} \times \omega_1$. Let $X_r = \{\alpha : (r, \alpha) \in X\}$ is either countable or size ω_1. When X_r is countable, one would like a bijection of ω with $\mathrm{ot}(X_r)$. This amounts to uniformizing a suitable relation $R \subseteq \mathbb{R} \times \mathrm{WO}$.

Cardinality of $\mathbb{R} \times \omega_1$

Fact (C-J-T)
Assume $\mathsf{AC}_\omega^\mathbb{R}$ *and all sets of reals have the perfect set property and the property of Baire.* $\mathbb{R} \times \omega_1$ *does not have* $\mathbb{R}$*-regularity or* ω_1*-regular cardinality.*

(AD^+) $\mathrm{gcof}(\mathbb{R} \times \omega_1) = \{X \in \mathrm{Surj}(\mathbb{R}) : |\mathbb{R}| \leq |X| \vee |\omega_1| \leq |X|\} = \{X \in \mathrm{Surj}(\mathbb{R}) : X \textit{ is uncountable}\}$.

Let $\Phi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ be $\Phi_1(r, \alpha) = r$ and $\Phi_2 : \mathbb{R} \times \omega_1 \to \omega_1$ be $\Phi_2(r, \alpha) = \alpha$.

Woodin observed that without Uniformization, there are many other cardinalities below $\mathbb{R} \times \omega_1$ than the four listed above. Work in $L(\mathbb{R}) \models \mathsf{AD}$. Let $\mathbb{X} = {}_\omega\mathbb{O}$ be the direct limit of the Vopěnka forcing on $\mathbb{R}^n$ for all $n \in \omega$.

Fact (Woodin; $\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$**)**
Let $W_2 = \bigsqcup_{r \in \mathbb{R}} \omega_2^{L[\mathbb{X}, r]}$. *Then* ω_1 *does not inject into* W_2 *and* $|\mathbb{R}| < |W_2| < |\mathbb{R} \times \omega_1|$.

Proof.
$\mathbb{X}$ has the property that for all function $\Phi : W_2 \to \mathbb{R}$, there is a $\mathbb{X}$-cone of e so that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$. Woodin showed there is a $\mathbb{X}$-cone of e so that $L[\mathbb{X}, e] \models \mathsf{CH}$. Suppose Φ is an injection. By picking an $e \in \mathbb{R}$ in a suitable $\mathbb{X}$-cone, $L[\mathbb{X}, e] \models \mathsf{CH}$, $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$, and $L[\mathbb{X}, e] \models (\Phi \cap L[\mathbb{X}, e]) : \{e\} \times \omega_2 \to \mathbb{R}$ is an injection. This is impossible. $\square$

There are no cardinalities between $|\mathbb{R}|$ and $|W_2|$.

Theorem (C-J; $\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$)
If $X \subseteq W_2$, then $|X| \leq |\mathbb{R}|$ or $|W_2| = |X|$.

Proof.
Let $X \subseteq W_2$. Consider the game H^2_X defined as follows.

$$
H^2_X \quad
\begin{array}{c|cccccccc}
1 & r_0 & & r_1 & & r_2 & & \cdots & r \\
2 & & x_0 & & x_1 & & x_2 & \cdots & x
\end{array}
$$

Player 2 wins H^X_2 if and only if $L[\mathbb{X}, r, x] \models \mathrm{ot}(X_r) < \omega_2^{L[\mathbb{X},r,x]}$. By AD, one of the two players has a winning strategy.

Player 2 has winning strategy: There is an injection of X into $\mathbb{R}$.

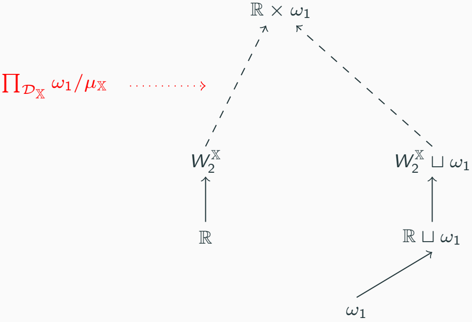
Player 1 has winning strategy: The winning stategy is used to find a useful $\mathbb{X}$-pointed tree from which an injection of W_2 into X is derived. □

Cardinality of $\mathbb{R} \times \omega_1$

Definition
Let $F : \mathbb{R} \to \omega_1$ be an $\mathbb{X}$-invariant function with respect to $\mathbb{X}$-constructibility degree. Let $W_F = \bigsqcup_{r \in \mathbb{R}} \omega_{F(r)}^{L[\mathbb{X},r]}$. For each $\alpha \in \prod_{\mathcal{D}_\mathbb{X}} \omega_1/\mu_\mathbb{X}$ (which is wellfounded under AD^+), let $Y_\alpha = |W_F^\mathbb{X}|$ for any F such that $[F]_{\mu_\mathbb{X}} = \alpha$.

Cardinality of $\mathbb{R} \times \omega_1$

Fact (C.-Jackson; $\mathsf{AD} + V = L(\mathbb{R})$)
$\langle Y_\alpha : \alpha \in \prod_{\mathcal{D}_{\mathbb{X}}} \omega_1/\mu_{\mathbb{X}} \rangle$, under the injection relation, is isomorphic to the ultrapower ordering. It is cofinal among the cardinalities below $\mathbb{R} \times \omega_1$ which do not possess a copy of ω_1. For any $\alpha \in \prod_{\mathcal{D}_{\mathbb{X}}} \omega_1/\mu_{\mathbb{X}}$, there are no cardinalities between Y_α and $Y_{\alpha+1}$. The first ω_1 many cardinalities below $|\mathbb{R} \times \omega_1|$ is exactly $\{Y_\alpha : \alpha < \omega_1\}$.



$L(\mathbb{R}) \models \mathsf{AD}$

Conjecture: This is the complete classification of the cardinalities below $\mathbb{R} \times \omega_1$ in $L(\mathbb{R})$.

Definition (Correct type functions)
Let $\epsilon \in \mathrm{ON}$ and $f : \epsilon \to \mathrm{ON}$. f is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\sup(f \restriction \alpha) < f(\alpha)$. f has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$ and $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$. f has the correct type if and only if f is both discontinuous everywhere and has uniform cofinality ω.

If $X \subseteq \mathrm{ON}$, then $[X]^\epsilon_*$ is the collection of increasing functions $f : \epsilon \to X$ of the correct type.

Definition (Correct type partition relation)
Let $\epsilon \leq \kappa$ and $\gamma < \kappa$. $\kappa \to_* (\kappa)^\epsilon_\gamma$ is the assertion that for all $P : [\kappa]^\epsilon_* \to \gamma$, there is a $\delta < \gamma$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $P(f) = \delta$.

$\kappa \to_* (\kappa)^{<\epsilon}_\gamma$ and $\kappa \to_* (\kappa)^\epsilon_{<\gamma}$ are given the obvious meanings.

If $\kappa \to_* (\kappa)^{<\kappa}_2$ (which implies $\kappa \to_* (\kappa)^{<\kappa}_{<\kappa}$), then κ is called a weak partition cardinal. If $\kappa \to_* (\kappa)^\kappa_2$, then κ is called a strong partition cardinal. If $\kappa \to_* (\kappa)^\kappa_{<\kappa}$, then κ is called a very strong partition cardinal.

Definition (Partition measures)
Let μ^ϵ_κ be a measure on $[\kappa]^\epsilon_*$ defined by $X \in \mu^\epsilon_\kappa$ if and only if there is a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq X$. $\kappa \to_* (\kappa)^\epsilon_2$ implies μ^ϵ_κ is an ultrafilter. $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$ implies μ^ϵ_κ is κ-complete. $\kappa \to_* (\kappa)^2_2$ implies the ω-club filter μ^1_κ is normal.

AD implies there are many partition cardinals.

Theorem

- *(Martin)* $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$.
- *(Martin-Paris)* $\omega_2 \to_* (\omega_2)^{<\omega_2}_2$.
- *(Jackson) For all* $n \in \omega$, $\boldsymbol{\delta}^1_{2n+1}$ *is a very strong partition cardinal.*
- *(Jackson; Kunen) For all* $n \in \omega$, $\boldsymbol{\delta}^1_{2n+2}$ *is a weak partition cardinal.*
- *(Kechris-Kleinberg-Moschovakis-Woodin)* $\boldsymbol{\delta}^2_1$ *and* Σ_1*-stable ordinals* $\boldsymbol{\delta}_A$ *of* $L(A, \mathbb{R})$ *for any* $A \in \mathscr{P}(\mathbb{R})$ *are very strong partition cardinals. There are cofinally many very strong partition cardinals below* Θ *(the supremum of the ordinals which are surjective images of* $\mathbb{R}$*).*

The Boldface GCH

Definition
The boldface GCH holds at κ if and only if there is injection of κ^+ into $\mathscr{P}(\kappa)$.

Fact
If there is a κ-complete ultrafilter on κ, then there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$.

If particular if $\kappa \to_* (\kappa)^2_2$ holds, then there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$.

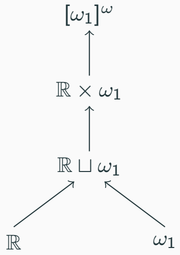
So $\omega_1 \to_* (\omega_1)^2_2$ implies the boldface GCH at ω.

Theorem (AD$^+$; Woodin, Steel)
The boldface GCH holds below Θ.

Martin showed that for $n \in \omega$, $\omega_{n+1} = \prod_{[\omega_1]^n_*} \omega_1/\mu^n_{\omega_1}$. With Jackson and Trang, we have a combinatorial proof of the boldface GCH below $\omega_{\omega+1}$. This type of argument should hold below the supremum of the projective ordinals and more generally to the extent of Jackson's descriptive analysis.

Cardinality of $[\omega_1]^\omega$

Using the $\omega_1 \rightarrow_* (\omega_1)^\omega_{<\omega_1}$, one has the following relation among the five familiar cardinalities below $|[\omega_1]^\omega|$.

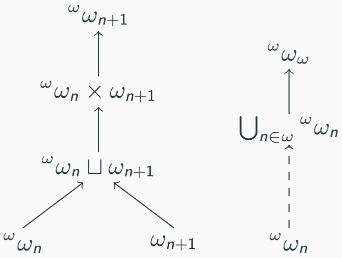


Theorem (Woodin; $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$)
If $X \subseteq [\omega_1]^\omega$, then either $|X| \leq |\mathbb{R} \times \omega_1|$ or $|[\omega_1]^\omega| = |X|$.

Thus under $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$, this is the complete classification of the cardinalities below $|[\omega_1]^\omega|$.

Cardinality of $[\omega_1]^\omega$

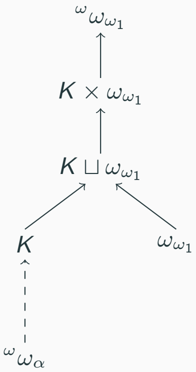
I have a proof of the Woodin $[\omega_1]^\omega$-dichotomy under $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ that can be adapted to ω-sequence through some higher cardinals.



A similar classification should hold for ${}^\omega\omega_\alpha$ when $\alpha < \omega_1$.

Cardinality of $[\omega_1]^\omega$

At ${}^\omega\omega_{\omega_1}$, new behaviors appear. Let $K = \bigsqcup_{w \in \mathrm{WO}} {}^\omega\omega_{\mathrm{ot}(w)}$. The following summarizes some of the structure below ${}^\omega(\omega_{\omega_1})$.



Conjecture: This is the complete classification below ${}^\omega\omega_{\omega_1}$ under $\mathrm{AD}_{\frac{1}{2}\mathbb{R}}$.

Cardinality of $[\omega_1]^\omega$

Fact
If $\kappa \to_* (\kappa)^2_2$, then for all $\epsilon < \kappa$, $[\kappa]^\epsilon$ is not κ-regular.

For example, $[\omega_1]^\omega = \bigcup_{\delta<\omega_1} [\delta]^\omega_*$ and $|[\delta]^\omega| = |\mathbb{R}| < |[\omega_1]^\omega|$. Thus $[\omega_1]^\omega$ is not locally regular.

Fact
If $\kappa \to (\kappa)^\epsilon_{<\kappa}$, then $[\kappa]^\epsilon$ is ${}^\mu\delta$ regular for all $\delta, \mu < \kappa$.

For example, $[\omega_1]^\omega$ is $\mathbb{R}$-regular.

Combined with Woodin's classification of $|[\omega_1]^\omega|$, one has the following local cofinality.

Fact ($\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$)
$\mathrm{lcof}([\omega_1]^\omega) = \{X : (\exists Z)(Z \subseteq [\omega_1]^\omega \wedge |Z| = |X| \wedge |\omega_1| \leq |X|\}$.

Definition
A set X is prime if and only if for all set Y and Z, if $|X| \leq |Y \times Z|$, then $|X| \leq |Y|$ or $|X| \leq |Z|$.

Theorem
Assume $\mathrm{AC}_\omega^\mathbb{R}$ and all sets of reals have the Baire property. Then $\mathbb{R}$ is prime.

Theorem (C-J-T)
Assume $\kappa \rightarrow_* (\kappa)_2^{\omega+\omega}$. Then $[\kappa]^\omega$ is prime.

Theorem (AD; C-J-T)
For all $\kappa \leq \omega_{\omega+1}$, $[\kappa]^\omega$ is prime.

Theorem (AD; C-J-T)
For all $n \in \omega$, $\mathscr{P}(\omega_{n+1})$ does not injection into $\mathscr{P}(\omega_n) \times \mathrm{ON}$.

Proof.
Suppose $\Phi : \mathscr{P}(\omega_3) \rightarrow \mathscr{P}(\omega_2) \times \mathrm{ON}$ is an injection. Then $\Phi : [\omega_3]^\omega \rightarrow \mathscr{P}(\omega_2) \times \mathrm{ON}$ is an injection. Since $[\omega_3]^\omega$ is prime, $|[\omega_3]^\omega| \leq |\mathscr{P}(\omega_2)|$ (impossible by boldface GCH at ω_2) or $|[\omega_3]^\omega|$ injects into an ordinal (which is impossible since $[\omega_3]^\omega$ is not wellorderable). $\square$

Theorem (Woodin; $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$**)**
If $X \subseteq [\omega_1]^\omega$, then either $|X| \leq |\mathbb{R} \times \omega_1|$ or $|[\omega_1]^\omega| = |X|$.

Woodin classification fails without uniformization.

Theorem (C.; $\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$**)**
Let $E_2 = \bigsqcup_{r \in \mathbb{R}} [\omega_2^{L[\mathbb{X},r]}]^\omega$. $\neg(|\omega_1| \leq |E_2|)$ and $\neg(|E_2| \leq |\mathbb{R} \times \omega_1|)$.

Proof.
Suppose $\Phi : E_2 \to \mathbb{R} \times \omega_1$ is an injection. There is a $\mathbb{X}$-cone of $e \in \mathbb{R}$ so that any inner model M with $\mathbb{X}, e \in M$, $\Phi \cap M \in M$. Let $f : \omega \to \omega_2^{L[\mathbb{X},e]}$ be the Namba generic function over $L[\mathbb{X}, e]$.
$\Phi \cap L[\mathbb{X}, e, f], \Phi^{-1} \cap L[\mathbb{X}, e, f] \in L[\mathbb{X}, e, f]$. Since $\Phi(e, f) \in L[\mathbb{X}, e]$ since Namba forcing adds no new reals. Since $(e, f) \in \Phi^{-1}(\Phi(e, f)) \in L[\mathbb{X}, e]$. It is impossible that the Namba generic belongs to the ground model. □

Woodin showed that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ under $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ indirectly using a certain subset of $[\omega_1]^{<\omega_1}$.

Definition (Woodin)
Let $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$.

Fact (AD^+; Woodin)
S_1 does not inject into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals. Thus $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

Proof.
Suppose $\Phi : S_1 \to {}^\omega\mathrm{ON}$ is an injection. Using the fact that all sets of reals have ∞-Borel codes, there is a set of ordinals J so that any inner model M with $J \in M$, one has $\Phi \cap M \in M$. Let $\xi < \omega_1^V$ be an inaccessible cardinal of $L[J]$. Let $G \subseteq \mathrm{Coll}(\omega, <\xi)$ be generic over $L[J]$. Let $f \in S_1$ be the generic function. $\Phi(f) \in L[J][G] \cap {}^\omega\mathrm{ON}$. Since every ω-sequence belongs to an initial segment of G, there is a $\delta < \xi$ so that $\Phi(f) \in L[J][G \restriction \delta]$. Since $\Phi \cap M$ is an injection in M, $f \in \Phi^{-1}(\Phi(f)) \in L[J][G \restriction \xi]$ which is impossible. $\square$

Cardinality of $[\omega_1]^{<\omega_1}$

To prove $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ in a manner that could generalize to higher cardinals, we investigate almost everywhere behavior or continuity properties of functions.

Let $\Phi : [\omega_1]^\omega \to \omega_1$ be defined by $\Phi(f) = \sup(f) + f(13) + f(7)$. Φ depends only on $\sup(f)$ and the 7th and 13th-value of f. The next results states all function have this behavior almost everywhere.

Theorem (AD**; C-J-T)**
Let $\epsilon < \omega_1$ and $\Phi : [\omega_1]^\epsilon \to \omega_1$.

(Short length continuity) There is a club $C \subseteq \omega_1$ and a $\delta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

(Strong short length continuity) There is a club $C \subseteq \omega_1$ and finitely many $\delta_0 < ... < \delta_{k-1} \leq \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if for all $i < k$, $\sup(f \restriction \delta_i) = \sup(g \restriction \delta_i)$, then $\Phi(f) = \Phi(g)$.

Cardinality of $[\omega_1]^{<\omega_1}$

Continuity yield very well controlled failure of injectiveness.

Theorem (C-J-T)
(AD) $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. $\neg(|[\omega_1]^{<\omega_1}| \leq |^\omega(\omega_\omega)|)$.

$(\mathsf{AD} + \mathsf{DC}_\mathbb{R})$ $^{<\omega_1}\omega_1$ *does not inject into* $^\omega\mathrm{ON}$, *the class of* ω*-sequences of ordinals.*

To extend these results to the familiar weak and strong partition cardinals of determinacy above ω_1 requires continuity results proved by pure partition properties.

Theorem (C-J-T)
(Countable cofinality short length continuity) Suppose $\epsilon < \kappa$, $\mathrm{cof}(\epsilon) = \omega$, $\kappa \to_* (\kappa)_2^{\epsilon \cdot \epsilon}$, *and* $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$. *Then there is a* $\delta < \epsilon$ *and a club* $C \subseteq \kappa$ *so that for all* $f, g \in [C]_*^\epsilon$, *if* $f \restriction \delta = g \restriction \delta$ *and* $\sup(f) = \sup(g)$, *then* $\Phi(f) = \Phi(g)$.

Theorem (C-J-T)
Suppose $\kappa \to_* (\kappa)_2^{<\kappa}$. *Then for all* $\lambda < \kappa$, $[\kappa]^{<\kappa}$ *does not inject into* ${}^\lambda\mathrm{ON}$, *the class of* λ*-sequences of ordinals.*

Cardinality of $[\omega_1]^{<\omega_1}$

Fact (C-J-T)
If $\kappa \rightarrow_* (\kappa)^{<\kappa}_2$, then $[\kappa]^{<\kappa}$ does not have κ-regular cardinality. Thus $[\kappa]^{<\kappa}$ does not have locally regular cardinality.

For example, $[\omega_1]^{<\omega_1} = \bigcup_{\epsilon<\omega_1}[\omega_1]^\epsilon$ and the previous theorem showed that $|[\omega_1|^\epsilon| < |[\omega_1]^{<\omega_1}|$.

Theorem (C-J-T)
Let $\kappa \rightarrow_* (\kappa)^\kappa_2$. For all $\lambda < \kappa$, $[\kappa]^{<\kappa}$ has λ-regular cardinality.

Theorem (C-J-T)
Let $\kappa \rightarrow_* (\kappa)^\kappa_{<\kappa}$. For all $\mu, \lambda < \kappa$, $[\kappa]^{<\kappa}$ is ${}^\mu\lambda$-regular.

Woodin showed that even under $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$, the structure of the cardinalities below $[\omega_1]^{<\omega_1}$ is very complicated and far from fully understood.

Cardinality of $\mathscr{P}(\omega_1)$

We have yet to encounter a candidate for another locally or even globally regular cardinality on the tame side of Hjorth dichotomy. A motivated goal for the study of the cardinality of $\mathscr{P}(\omega_1)$ is the following conjecture.

Conjecture: $\mathscr{P}(\omega_1)$ has locally regular cardinality and even globally regular cardinality.

We will provide emperical evidence for this conjecture by searching for cofinality of $\mathscr{P}(\omega_1)$.

Theorem (AD**)**
$|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$.

Proof.
Suppose $\Phi : \mathscr{P}(\omega_1) \to [\omega_1]^{<\omega_1}$ is an injection. $L[\Phi] \models$ ZFC and $L[\Phi] \models \omega_1^V$ is inaccessible. $L[\Phi] \models |\mathscr{P}(\omega_1^V)| > \omega_1^V$ and $|[\omega_1{}^V]^{<\omega_1{}^V}| = \omega_1^V$. Also $L[\Phi] \models \Phi : \mathscr{P}(\omega_1^V) \to [\omega_1^V]^{<\omega_1^V}$ is an injection, which is impossible. □

This proof is not ideal with respect to the regularity conjecture as its does verify any instance of regularity for $\mathscr{P}(\omega_1)$. The regularity conjecture should be used as a standard for all cardinality computation relative to $\mathscr{P}(\omega_1)$.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (AD; C-J)
(Almost everywhere continuity) Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$*. There is a club* $C \subseteq \omega_1$ *so that for all* $f \in [C]^{\omega_1}_*$*, there exists an* $\alpha < \omega_1$ *so that for all* $g \in [C]^{\omega_1}_*$*, if* $g \restriction \alpha = f \restriction \alpha$*, then* $\Phi(f) = \Phi(g)$*.*

Theorem (AD; C-J)
$\mathscr{P}(\omega_1)$ *has* ω_1*-regular cardinality and in fact, has* $[\omega_1]^{<\omega_1}$*-regular cardinality. Thus* $|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$*.*

Definition (Short length club uniformization)
Almost everywhere short length club uniformization holds at κ if and only if for all relations $R \subseteq [\kappa]^{<\kappa}_* \times \text{club}_\kappa$ which is $\subseteq$-downward closed in the club$_\kappa$-coordinate, there is a club $D \subseteq \kappa$ and a function $\Lambda : [D]^{<\kappa}_* \cap \text{dom}(R) \to \text{club}_\kappa$ so that for all $\ell \in [D]^{<\kappa}_* \cap \text{dom}(R)$, $R(\ell, \Lambda(\ell))$.

Originally this property was established for ω_1 using Kechris-Woodin generic coding which does not generalize. I showed this holds for any cardinal which is very reasonable in the sense that it possesses stronger version of Martin's good coding system for ${}^\kappa\kappa$ called a good coding family which also code bounded sequences. All the familar reasonable cardinals (possessing Martin good coding systems) are very reasonable.

Theorem (C.)
If $\kappa \to_* (\kappa)^\kappa_2$ and the almost everywhere short length club uniformization holds at κ, then every function $\Phi : [\kappa]^\kappa_* \to \kappa$ is continuous μ^κ_κ-almost everywhere (in the earlier sense for ω_1).

Ultrapower of the Strong Partition Measure on ω_1

Continuity for functions from $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ have applications beyond cardinality computation and was originally motivated investigation of the ultrapowers of the strong partition measure.

Fact (Kechris-Kleinberg-Moschovakis-Woodin; AD**)**
There are cofinally many strong partition cardinals below Θ*.*

Fact (Kechris-Woodin)
Assume $V = L(\mathbb{R})$*. If there are cofinally many strong partition cardinals below* Θ*, then* AD *holds.*

Woodin asked if $V = L(\mathbb{R})$ and $\mathsf{DC}_{\mathbb{R}}$ holds, then does $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, then does AD holds?

If the conjecture is true, then the strong partition property on ω_1 can generate many other strong partition cardinals. How can this be possible? The most obvious attempt is to take ultrapowers of the strong partition cardinal by the strong partition measure. Since AD is the only universe for which we know there are strong partition cardinals, we may as well investigate these ultrapowers under AD.

Ultrapower of the Strong Partition Measure on ω_1

Fact (AD; Martin)
$\omega_2 = \prod_{\omega_1} \omega_1/\mu^1_{\omega_1}$ *and* ω_2 *is a weak partition cardinal.*

So a weak partition cardinal can be obtained by taking ultrapowers. Under AD, Jackson showed the next strong partition cardinal is $\omega_{\omega+1} = \boldsymbol{\delta}^1_3$.

Goldberg and Henle asked if $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} < \omega_{\omega+1}$.

Theorem (AD; C)
$\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} < \omega_{\omega+1}$.

Main ideas: This bound would follow from the Kunen-Martin theorem is we can code the ultrapower by a $\boldsymbol{\Sigma}^1_3$ and hence ω_ω-Suslin well founded relation. The representatives are functions $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. The collection of good codes (in the sense of Martin) for elements of ${}^{\omega_1}\omega_1$ is a complete $\boldsymbol{\Pi}^1_2$ set. The natural ultrapower relation requires a universal quantification of this set which is too complicated. Instead one would like to use continuity to quantify over over continuity points which are bounded sequences. The proof of the short length club uniformization using good coding families provides necessary coding mechanism.

The spirit of the Goldberg-Henle question is that an ultrapower construction below the second strong partition cardinal, $\omega_{\omega+1}$, should not be able to exceed $\omega_{\omega+1}$.

Conjecture: Is $\prod_{[\omega_1]^{\omega_1}_*} \omega_{\omega+1}/\mu^{\omega_1}_{\omega_1} = \omega_{\omega+1}$? In particular, is $\prod_{[\omega_1]^{\omega_1}_*} \omega_2/\mu^{\omega_1}_{\omega_1} < \omega_{\omega+1}$?

Theorem (AD**; C)**
If $\epsilon < \omega_1$, $\prod_{[\omega_1]^{\epsilon}_*} \omega_{\omega+1}/\mu^{\epsilon}_{\omega_1} = \omega_{\omega+1}$.

To answer the above questions, one needs an adequate understanding of the almost everywhere behavior of functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_2$. I do not know what suitable property could be. However, we do know very weak continuity results which are sufficient for addressing issues of regularity for $\mathscr{P}(\omega_1)$.

Theorem (C)
Assume $\kappa \to_* (\kappa)^\kappa_2$. Then $\mathscr{P}(\kappa)$ has ON-regular cardinality.

Consider $\Psi : [\omega_1]^{\omega_1}_* \to \omega_2$ defined by $\Psi(f) = [f]_{\mu^1_{\omega_1}}$, which is the ultrapower of f under the club measure on ω_1. Note that if $A \subseteq \omega_1$ has measure 0 according to the club measure, then for any $f, g \in [\omega_1]^{\omega_1}_*$ with $f \restriction \kappa \setminus A = g \restriction \kappa \setminus A$, then $\Phi(f) = \Phi(g)$. This motivates a weak continuity result.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (C-J-T)
Assume $\kappa \to_* (\kappa)^\kappa_{<\kappa}$*. Let* $\langle A_\alpha : \alpha < \kappa \rangle$ *be a sequence of disjoint subsets of* κ *so that each* $\kappa \setminus A_\alpha$ *is unbounded in* κ*. Let* $\Phi : [\kappa]^\kappa \to \mathrm{ON}$*. Then there is a finite set* F *and a club* $C \subseteq \kappa$ *so that for all* $\alpha \notin F$*, there is a* $\xi < \kappa$ *so that for all* $f, g \in [C]^\kappa_*$*, if* $f(0) > \xi$*,* $g(0) > \xi$ *and* $f \restriction \kappa \setminus A_\alpha = g \restriction \kappa \setminus A_\alpha$*, then* $\Phi(f) = \Phi(g)$*.*

Theorem (C-J-T)
Suppose $\kappa \to_* (\kappa)^\kappa_{<\kappa}$*. Then* $\mathscr{P}(\kappa)$ *is* ${}^{<\kappa}\mathrm{ON}$*-regular.*

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (AD; **C-J**)
$\mathscr{P}(\omega_1)$ has ω_1-regular cardinality and in fact, has $[\omega_1]^{<\omega_1}$-regular cardinality. Thus $|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$.

Thus $\mathscr{P}(\omega_1)$ would be locally regular if the following conjecture has a positive answer.

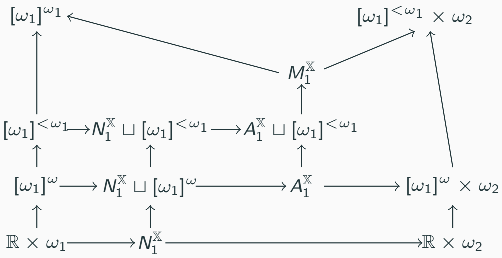
Conjecture: Assume $\mathsf{AD}_{\mathbb{R}}$. Suppose $X \subseteq \mathscr{P}(\omega_1)$. Then $|X| \leq |[\omega_1]^{<\omega_1}|$ or $|X| = |\mathscr{P}(\omega_1)|$.

Cardinality of $\mathscr{P}(\omega_1)$

Conjecture: Assume $\mathsf{AD}_{\mathbb{R}}$. Suppose $X \subseteq \mathscr{P}(\omega_1)$. Then $|X| \leq |[\omega_1]^{<\omega_1}|$ or $|X| = |\mathscr{P}(\omega_1)|$.

This conjecture cannot be true if the assumption of $\mathsf{AD}_{\mathbb{R}}$ is dropped.

Work in $L(\mathbb{R})$. Let $N_1^{\mathbb{X}} = \bigsqcup_{r \in \mathbb{R}} (\omega_1^V)^{+L[\mathbb{X},r]}$, $A_1^{\mathbb{X}} = \bigsqcup_{f \in [\omega_1]^\omega} (\omega_1^V)^{+L[\mathbb{X},f]}$, and $M_1^{\mathbb{X}} = \bigsqcup_{f \in [\omega_1]^{<\omega_1}} (\omega_1^V)^{+L[\mathbb{X},f]}$.



In this setting, the local regularity of $\mathscr{P}(\omega_1)$ would follow from the following conjecture.

Conjecture: $(\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R}))$ If $X \subseteq \mathscr{P}(\omega_1)$, then $|X| \leq |M_1^{\mathbb{X}}|$ or $|X| = |\mathscr{P}(\omega_1)|$.

Many familiar sets which are surjective images of $\mathbb{R}$ are naturally presented as quotients of equivalence relations on $\mathbb{R}$.

Fact (Woodin; AD^+**)**
If $A \subseteq \mathbb{R}$ is a nonempty countable set with an ∞-Borel code (S, φ). Then A has an $\mathrm{OD}_{\{S\}}$ member and in fact $A \subseteq \mathrm{HOD}_{\{S\}}$.

Fact (C; AD^+**)**
Let E be E_0, E_1, E_2, countable Borel, essentially countable, hyperfinite, smooth, or hypersmooth (more generally, is a pinned analytic equivalence relation over models of ZFC*, in the sense of Zapletal). If A is an E-class with ∞-Borel code (S, φ), then A has an $\mathrm{OD}_{\{S\}}$-member.*

Martin developed the notion of a good coding system for ${}^{\epsilon}\kappa$ by reals. The existence of a good coding system for ${}^{\omega\cdot\epsilon}\kappa$ yields the partition relation $\kappa \rightarrow_* (\kappa)^{\epsilon}_{<\kappa}$.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (AD; C-J-T)
Suppose κ is a cardinal and $\epsilon \leq \kappa$. Let $M \models \mathsf{AD}^+$ be an inner model with $\mathbb{R} \subseteq M$ and M has a good coding system for ${}^{\omega \cdot \epsilon}\kappa$. Let E be one of the following equivalence relations.

- *E is an equivalence relation with all countable classes (E does not need to belong to M).*
- *E is E_0, E_1, E_2, countable, essentially countable, hyperfinite, smooth, or hypersmooth.*

Then $[\kappa]^{\epsilon}_{*}$ has $\mathbb{R}/E$-regular cardinality.

Kechris showed AD implies $L(\mathbb{R}) \models \mathsf{AD}^+$.

Theorem (AD; C-J-T)
$\mathscr{P}(\omega_1)$ has $\mathbb{R}/E$-regular cardinality when E is one of the above equivalence relations.

$=^+$ does not fall into this setting. Note $\mathbb{R}/=^+ \approx \mathscr{P}_{\omega_1}(\mathbb{R})$. However, we can still show the following.

Theorem (AD; C-J-T)
$\mathscr{P}(\omega_1)$ has $\mathscr{P}_{\omega_1}(\mathbb{R})$-regular cardinality.

Questions

- Find another locally regular or globally regular cardinality. Is local regularity and global regularity always the same concept?
- Calibrate the cofinality of $\mathscr{P}(\omega_1)$. Is $\mathscr{P}(\omega_1)$ locally regular or globally regular?
- Is $\mathscr{P}(\omega_2)$ even 2-regular under any determinacy assumption?

Thanks for listening!