Clopen type semigroups of actions on zero-dimensional compact spaces

J. Melleray

Caltech Logic seminar, Feb. 28, 2023

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While this appears largely out of reach of current methods, a variant of this might be more tractable:

•Let G be countable amenable. Given a minimal Cantor G-action α, does there exist a minimal Cantor $\mathbb{Z}$-action which preserves the same Borel probability measures as α?

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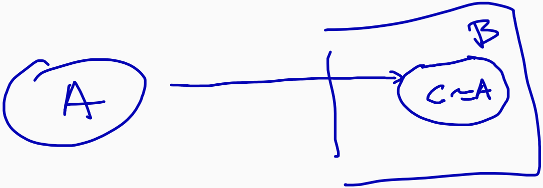

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The last one is also necessary, because of a result of Glasner–Weiss using properties of the *topological full group*.

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Given an action $\alpha\colon G \curvearrowright X$, the *topological full group* $[[\alpha]]$ consists of all homeomorphisms γ for which there exists a clopen partition $(U_i)_{i\in I}$ s.t. γ coincides with some $g_i \in G$ on each U_i.

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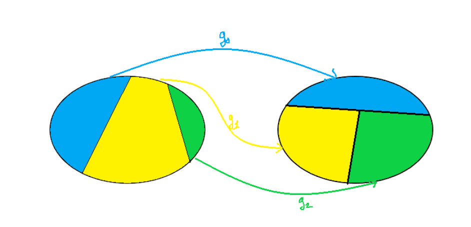

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Definition (Buck 2013, Kerr 2017)
$\alpha\colon G \curvearrowright X$ has *dynamical comparison* if :

$$\forall A, B \in \mathrm{Clopen}(X)^* \, (\forall \mu \in \mathbb{P}(\alpha)\, \mu(A) < \mu(B)) \Rightarrow \exists \gamma \in [[\alpha]]\, \gamma A \subset B$$

nonempty

$(\forall \mu \in \mathbb{P}(\alpha) \;\; \mu(\gamma A) = \mu(A))$

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- Free actions of elementary amenable groups (Kerr-Narishkyn 2021).

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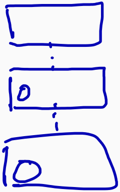

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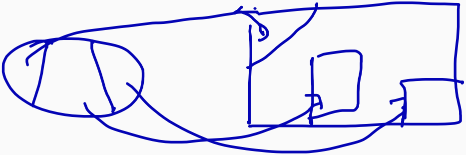
finite support



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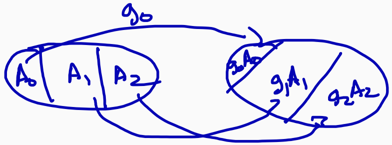
Then set $[A] + [B] = [\tilde{A} \sqcup \tilde{B}]$ where $[\tilde{A}] = [A]$, $[\tilde{B}] = [B]$ and $\tilde{A} \cap \tilde{B} = \emptyset$. .



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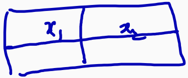
We obtain a commutative monoid which we denote $T(\alpha)$. It is a *refinement monoid*: if $x_1 + x_2 = y_1 + y_2$ then there exist $z_{i,j}$ such that $x_i = z_{i,1} + z_{i,2}$ and $y_j = z_{1,j} + z_{2,j}$ for $i, j \in \{1, 2\}$.



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We order $T(\alpha)$ using the algebraic preorder:

$\mu(a) + \mu(c) = \mu(b)$, $\mu(c) \geq 0$

$$(a \leq b) \Leftrightarrow (\exists c \;\; a + c = b)$$

States and invariant measures

Definition
A *state* is a morphism $\mu\colon (T(\alpha), +) \to [0, +\infty]$. It is *normalized* if $\mu([X]) = 1$. I denote by $\mathbb{P}(\alpha)$ the set of normalized states.

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Theorem (Tarski)
Given $a \in T(\alpha)$, there exists a state μ such that $\mu(a) = 1$ iff $(n+1)a \not\leq na$ for all $n \in \mathbb{N}$.

$(n+1) \leq n$

Dynamical comparison seen in $T(\alpha)$

Proposition
$\alpha\colon G \curvearrowright X$ has dynamical comparison iff

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$T(\alpha)$ is *almost unperforated* if

$$\forall a, b \in T(\alpha) \, \forall n \in \mathbb{N} \quad (n+1)a \leq nb \Rightarrow a \leq b$$

Dynamical comparison and almost unperforation

Proposition

$\forall a \exists n \; a \leq nb$

- Assume that G is amenable. Then α has dynamical comparison iff for every order unit $b \in T(\alpha)$ and every a such that $(n+1)a \leq nb$ for some n, one has $a \leq b$.

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- If α is minimal then α has dynamical comparison iff $T(\alpha)$ is almost unperforated (and if $\mathbb{P}(\alpha) = \emptyset$ then $a \leq b$ for all nonzero $a, b \in T(\alpha)$).

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This statement has a precursor in Kerr (2018), a version valid for all compact metrizable spaces is given by Ma (2019) and there is a related statement for second-countable ample groupoids by Ara–Bönicke–Bosa–Li (2020). Kerr was the first to notice the connection with clopen type semigroups.

Weak comparability and cancellativity

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Proposition (M.)

- α has dynamical comparison iff $T(\alpha)$ has the *weak comparability property*:

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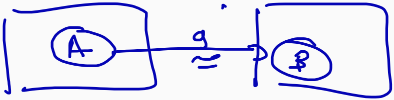

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This follows from work of Ara–Pardo (1996) and Ara–Goodearl–Pardo–Tyukavkin (1995) on refinement monoids. (and I do not know of a more direct proof that dynamical comparison and minimality imply cancellativity !)

Existence of a dense locally finite subgroup in $[[\alpha]]$

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Assume that $\alpha \colon G \curvearrowright X$ is a minimal action of G on a Cantor space such that $\mathbb{P}(\alpha) \neq \emptyset$. Then $[[\alpha]]$ has a dense, locally finite subgroup iff $T(\alpha)$ is unperforated.

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For minimal $\mathbb{Z}$-actions certain dense locally finite subgroups of $[[\alpha]]$ play a key role in the Giordano–Putnam–Skau classification.

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• If $[[\alpha]]$ has a dense, locally finite subgroup then it is easy to see that $T(\alpha)$ is unperforated.

• For the converse, use a result of Ara–Goodearl (2015) which shows (assuming unperforation) that $T(\alpha)$ is an inductive limit of finitely generated refinement monoids (their generators can then be used to build the finite groups we are looking for).

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This gives a surjective homomorphism from $T(\alpha)$ to the positive cone $H(\alpha)^+$; it is injective iff $T(\alpha)$ is cancellative (actually $H(\alpha)$ is the Grothendieck group of $T(\alpha)$).

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Matui gave examples of free minimal Cantor actions α of $\mathbb{Z}^2$ in which $H(\alpha)$ has torsion. For such actions $T(\alpha)$ cannot be unperforated, so $[[\alpha]]$ does not have a dense locally finite group.

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Hence $G \curvearrowright \beta G$ has dynamical comparison when G is amenable : if $A, B \subset G$ are s.t. $\mu(A) < \mu(B)$ for any G-invariant f.a.p.m, there exist $A_1, \dots A_n$, $g_1, \dots, g_n$ s.t.

$$\bigsqcup_{i=1}^{n} A_i = A \text{ and } \bigsqcup_{i=1}^{n} g_i A_i \subset B$$

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$T(G \curvearrowright \mu G)$ is isomorphic to a submonoid of $T(G \curvearrowright \beta G)$. Hence it is unperforated and $\leq$ is a partial order on $T(G \curvearrowright \mu G)$.

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For amenable G, the fact that any minimal action is a factor of a free minimal action with dynamical comparison also follows from work of Conley–Jackson–Kerr–Marks–Seward–Tucker-Drob (2017)

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Minimal actions form a G_δ subset $\mathrm{Min}(G) \subset \mathrm{Act}(G)$.

$\forall U$ clopen $\neq \emptyset$ $\quad \underbrace{\exists g_1 \dots g_n \in G \quad \bigcup_{i=1}^{n} g_i U = X}_{\text{open}}$

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6. Unique ergodicity.

Some questions on $\mathrm{Min}(G)$ and $\mathrm{Act}(G)$

For the questions I focus on the minimal Cantor case.

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- For amenable G, is unique ergodicity generic in $\mathrm{Min}(G)$?
- Does every countable group admit a uniquely ergodic, free Cantor action? True for amenable G (Rosenthal 1985) Every group admits a free, minimal Cantor action with an invariant probability measure (Elek 2020).

Thank you for your attention!