Classification strength of Polish groups and involving S_∞

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This work was done while at:

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2. Hebrew University of Jerusalem, hosted by Omer Ben-Neria
3. University of Toronto, hosted by Spencer Unger

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Classification Strength

Invariant Descriptive Set Theory

Invariant descriptive set theory, the theory of definable equivalence relations, has (at least) two main objectives.

Measure and compare the difficulties of classification problems in mathematics

1. Examples: graph isomorphism, isomorphism problem in ergodic theory
2. Compare classification problems via *definable* reductions
3. Allows one to determine if a classification problem has a "satisfactory" solution

Classification Strength

Invariant Descriptive Set Theory

Invariant descriptive set theory, the theory of definable equivalence relations, has (at least) two main objectives.

Study "definable" cardinality

1. Under AC, cardinals are linearly-ordered
2. Without choice, picture much more complicated, and requires difficult set theory to study
3. Interesting playground: consider quotients X/E of nice topological spaces by definable equivalence relations, and definable bijections between them.

Classification Strength

Invariant Descriptive Set Theory

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3. Important special case: the Polish group is *non-Archimedean*
   3.1 equivalently, a closed subgroup of S_∞
   3.2 equivalently, the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a countable structure in a countable language.
4. Notion of definable for a reduction will almost always be a Borel function.

Classification Strength

Invariant Descriptive Set Theory

We will follow the following conventions:

1. G and H are Polish groups
2. X and Y are Polish spaces
3. Orbit equivalence relations E_X^G and E_Y^H are induced by continuous actions
4. $\mathcal{M}$ is a countable structure in a countable relational language, and is ultrahomogeneous

Classification Strength

Definition 1.1

Given equivalence relations E and F on Polish spaces X and Y, a Borel reduction from E to F is a Borel function $f : X \to Y$ such that $x\ E\ y$ iff $f(x)\ F\ f(y)$. When such a reduction exists, we write $E \leq_B F$.

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We say that G has stronger classification strength than H, denoted $H \preceq_{CS} G$, iff G classifies every orbit equivalence relation induced by H.

Classification Strength

Invariant Descriptive Set Theory

- $\preceq_{CS}$ defines a preorder on the class of all Polish groups;
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There are two special equivalence relations we will need to consider:

1. $=^+$ lives on $\mathbb{R}^\omega$ where $(x_n)_{n\in\omega} =^+ (y_n)_{n\in\omega}$ iff $\{x_n \mid n \in \omega\} = \{y_n \mid n \in \omega\}$;
2. E_{ω_1} lives on LO, the G_δ subset of elements of $2^{\omega\times\omega}$ which codes linear orders on ω, where $x \; E_{\omega_1} \; y$ iff both $<_x$ and $<_y$ are ill-founded, or if they are both well-ordered with the same ordertype.

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Say $E_{\omega_1} \leq_{aB} F$ if there is a Borel function which is a reduction except for the single proper analytic class of non-wellorders.

Definition 1.2

*Say that G **involves** H iff there is a closed subgroup G' of G and a continuous surjective homomorphism from G' onto H.*

Proposition 1 (Mackey, Hjorth)

If G involves H then $H \preceq_{CS} G$.

The converse is easily false, as the trivial group does not involve any nontrivial compact Polish groups (such as $\mathbb{Z}_2^\omega$), yet every compact Polish group is below the trivial group in classification strength.

Classification Strength

Cli groups

A Polish group G is cli iff it has a complete left-invariant compatible metric.

Theorem (Hjorth)

If G is cli then it does not classify $=^+$

So if G is cli then G is not above S_∞ in classification strength.

Classification Strength

Cli groups

Theorem

For $G = Aut(\mathcal{M})$, then TFAE

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Theorem

For $G = Aut(\mathcal{M})$*, then TFAE*

1. G *cli;*
2. *(Gao) there is no non-trivial* $\mathcal{L}_{\omega_1,\omega}$*-embedding of* $\mathcal{M}$ *into* $\mathcal{M}$*;*
3. *(Gao) there is no uncountable model of the Scott sentence of* $\mathcal{M}$*;*
4. *(Gao)* $\mathcal{M}$ *is the quasi-definable closure of* $\emptyset$*;*
5. *(Deissler)* $\mathrm{Drk}(a, \emptyset) < \infty$ *for every* $a \in M$*;*
6. *(Deissler)* $\mathrm{Drk}(a, \emptyset) < \omega_1$ *for every* $a \in M$*;*
7. *(Gao, Hjorth, Thompson)* G *has the pinned property in any model of ZFC;*

Classification Strength

Cli groups

This result has been extended to general Polish groups in several ways (won't discuss)

Together with Aristotelis, we are exploring a hierarchy of cli Polish groups and showing that it is strictly-increasing with respect to classification strength.

A hierarchy of groups below a the automorphism group of a "universal group tree", strictly increasing with respect to classification strength, studied by Clemens-Coskey.

INVOLVING S_∞

S_∞ is the Polish group of permutations of a countably-infinite set.

A Polish group is non-Archimedean iff it is involved by S_∞. But what about Polish groups involving S_∞?

Two previously-known sufficient conditions

1. (Baldwin-Friedman-Koerwien-Laskowski) If Age($\mathcal{M}$) satisfies disjoint/strong amalgamation, then Aut($\mathcal{M}$) involves S_∞;
2. (Hjorth) For any Polish group G, if $E_{\omega_1} \leq_{aB} E_X^G$ then G involves S_∞.

Recall:

Proposition 2 (Mackey, Hjorth)

If G involves H then $H \preceq_{CS} G$.

Perhaps we can find a “weak" converse?

Perhaps $H \preceq_{CS} G$ implies that G involves the quotient of H by a "small" normal subgroup, or G involves a "large" subgroup of H?

This seems unlikely, and would trivialize a lot of work in the field of invariant descriptive set theory.

However, we will see that it is true in the case that H is S_∞!

INVOLVING S_∞

MAIN THEOREM

Theorem (A.)

Let $G = Aut(\mathcal{M})$ be a non-Archimedean Polish group. Then TFAE

1. $S_\infty \preceq_{CS} G$

INVOLVING S_∞

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Mackey, Hjorth implies $(2) \to (1)$, $(1) \to (7)$ is by definition, the rest is new

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7. *G classifies* $=^+$;

By a result of Hjorth, we can also add

8. G induces an orbit equivalence relation with arbitrarily large virtual classes

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Using the same Hjorth result, and a result of Larson-Zapletal, we can add

9. G has the unpinned property in the Solovay model.

Involving S_∞

Classifies $=^+$ implies nontrivial indiscernible support function

We use the orbit continuity lemma:

Theorem (Hjorth, Lupini-Panagiotopoulos)

Let E_X^G and E_Y^H be orbit equivalence relations and $f : X \to Y$ a Baire-measurable homomorphism. Let $G_0 \leq G$ be a countable dense subgroup. Then there is a comeager subset $C \subseteq X$ satisfying

1. *f is continuous on C;*
2. *for every $x \in C$, there is a comeager set of $g \in G$ such that $g \cdot x \in C$;*
3. *for every $x \in C$ and $g \in G_0$, $g \cdot x \in C$;*
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4. *for every $x_0 \in C$ and $g \in G_0$ and open $V \subseteq H$ such that $f(g \cdot x_0) \in V \cdot f(x_0)$, there is $W \ni g$ open such that for a comeager set of $w \in W$, $f(w \cdot x_0) \in V \cdot f(x_0)$.*

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4. *for every $x_0 \in C$ and $g \in G_0$ and open $V \subseteq H$ such that $f(g \cdot x_0) \in V \cdot f(x_0)$, there is $U \ni x_0$ and $W \ni g$ open such that for every $x \in U \cap C$ and for a comeager set of $w \in W$, $f(w \cdot x) \in V \cdot f(x)$.*

INVOLVING S_∞

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A support function of M is a function

$$\mathrm{supp} : \mathcal{P}_{\text{fin}}(M) \to \mathcal{P}_{\text{fin}}(\omega)$$

satisfying

- $\mathrm{supp}(\emptyset) = \emptyset$;
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Involving S_∞

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and indiscernible iff

- for any $A \subseteq B$ and $u \subseteq v$ with $\mathrm{supp}(A) = u$ and $\mathrm{supp}(B) = v$, if $w \cong_u v$ then there is some $C \cong_A B$ with $\mathrm{supp}(C) = w$.

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Here, $C \cong_A B$ means there is an automorphism π of $\mathcal{M}$ satisfying $\pi[C] = B$ with $\pi \restriction A = \mathrm{id}_A$, and $w \cong_u v$ simply means $|w \setminus u| = |v \setminus u|$.

INVOLVING S_∞

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- Let A be a countably-infinite set and Δ a countably-infinite group and $\alpha : \Delta \curvearrowright A$ a free action.
- Let $Q = \mathrm{Aut}(A, \alpha)$ be the non-Archimedean Polish group of permutations π of A satisfying $\pi(\delta \cdot a) = \delta \cdot \pi(a)$ for every $a \in A$ and $\delta \in \Delta$.

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- Let Z be the Polish space $\mathbb{R}^A$ with the natural action $Q \curvearrowright Z$.

- It's straightforward to check $E_Z^Q \sim_B =^+$

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- Recall $Q = \mathrm{Aut}(A, \alpha)$ the non-Archimedean Polish group of permutations π of A satisfying $\pi(\delta \cdot a) = \delta \cdot \pi(a)$ for every $a \in A$ and $\delta \in \Delta$.
- Let $T \subseteq A$ intersect every Δ-orbit exactly once.
- Then the sets $\mathrm{Stab}_u(Q)$ for every finite $u \subseteq T$ form a countable local basis of the identity of Q.

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(B) Let $C \subseteq X$ be as in the orbit continuity lemma, and fix $x_0 \in C$;

(C) Say that $u \subseteq T$ supports $A \subseteq M$ iff there is an open neighborhood $U \ni x_0$ such that for every $x \in U \cap C$, $f(wx) \in \mathrm{Stab}_A(G) \cdot f(x)$;

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(C) Say that $u \subseteq T$ supports $A \subseteq M$ iff there is an open neighborhood $U \ni x_0$ such that for every $x \in U \cap C$, $f(wx) \in \mathrm{Stab}_A(G) \cdot f(x)$;

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Involving S_∞

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Involving S_∞

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- In $V(C)$, for any finite $u, v, w \subseteq C$ with $u \subseteq v, w$ and $v \cong_u w$, both v and w have the same type over $V(u)$.
- (In the paper, this is instead formalized in terms of generic ergodicity / density of orbits)

INVOLVING S_∞

NONTRIVIAL INDISCERNIBLE SUPPORT FUNCTION IMPLIES NON-ORDINAL RANK

Before we define the Krk, we will first motivate it.

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Theorem (Knight)

There exists a countable structure $\mathcal{K} = (K, <, f_n)_{n\in\omega}$ satisfying

1. *$(K, <)$ is a linear order;*
2. *for every $a \in K$, $\{b \in K \mid b < a\} = \{f_n(a) \mid n \in \omega\}$; and*
3. *there is a nontrivial $\mathcal{L}_{\omega_1,\omega}$-elementary embedding of $\mathcal{K}$ into $\mathcal{K}$.*

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The structure $\mathcal{K}$ is referred to as "Knight's model", though the construction is not unique.

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Proposition 3

1. *(Gao) $Aut(\mathcal{K})$ is not cli*
2. *(Hjorth) $Aut(\mathcal{K})$ does not involve S_∞ and in fact does not classify $=^+$*

INVOLVING S_∞

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The automorphism group of Knight's model was essentially the only known example of a Polish group which is not cli but doesn't involve S_∞.

Involving S_∞

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Involving S_∞

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As a philosophical corollary of our results, Knight's model is basically the simplest such group.

Given a countable structure $\mathcal{M}$ (assumed to be ultrahomogeneous), define $\mathrm{Krk}(a, \bar{b})$ for $a \in M$ and $\bar{b} \in M^{<\omega}$ as follows:

1. $\mathrm{Krk}(a, \bar{b}) \leq 0$ iff for every $a' \cong_{\bar{b}} a$, $a' = a$;
2. $\mathrm{Krk}(a, \bar{b}) \leq \alpha$ iff either
   - 2.1 there is some c such that for every $c' \cong_{\bar{b}} c$, $\mathrm{Krk}(a, c'\bar{b}) < \alpha$; or
   - 2.2 for every $a' \cong_{\bar{b}} a$, either $\mathrm{Krk}(a', a\bar{b}) < \alpha$ or $\mathrm{Krk}(a, a'\bar{b}) < \alpha$
3. $\mathrm{Krk}(a, \bar{b}) = \infty$ iff $\mathrm{Krk}(a, \bar{b}) > \alpha$ for every ordinal α.

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NONTRIVIAL INDISCERNIBLE SUPPORT FUNCTION IMPLIES NON-ORDINAL RANK

1. If supp is a nontrivial indiscernible support function on $\mathcal{M}$ and $\mathrm{Krk}(a, \bar{b}) < \infty$, then $\mathrm{supp}(a\bar{b}) \subseteq \mathrm{supp}(\bar{b})$.

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3. If supp is nontrivial, then there must be some $a \in M$ such that $\mathrm{supp}(a) \supsetneq \mathrm{supp}(\emptyset)$ (small argument).

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4. Then by contrapositive of lemma, $\mathrm{Krk}(a, \emptyset) = \infty$.
5. By the usual arguments, if a countable structure has ordinal Krk, it should be countable.

INVOLVING S_∞

ORDINAL RANK IMPLIES DISJOINT AMALGAMATION RELATIVE TO CL

Given an ultrahomogeneous structure $\mathcal{M}$, a function

$$\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$$

is a closure operator iff

1. $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;
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and finitary iff

6. $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) \mid A_0 \subseteq A \text{ finite}\}$.

INVOLVING S_∞

ORDINAL RANK IMPLIES DISJOINT AMALGAMATION RELATIVE TO CL

Given such cl, by its automorphism-invariance, we may view it as a family of closure operators $\mathrm{cl}_{\mathcal{A}}$ for each $\mathcal{A} \in \mathrm{Age}(\mathcal{M})$ satisfying

$$\mathrm{cl}_{\mathcal{A}}(x) = \mathrm{cl}_{\mathcal{B}}(x) \cap A$$

for every $\mathcal{A} \leq \mathcal{B}$ in $\mathrm{Age}(\mathcal{M})$ and $x \subseteq A$.

Involving S_∞

Ordinal rank implies disjoint amalgamation relative to cl

We say $\mathrm{Age}(\mathcal{M})$ satisfies the disjoint property relative to cl iff for every $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Age}(\mathcal{M})$ with $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{A} \leq \mathcal{C}$, there is some $\mathcal{C}' \cong_{\mathcal{A}} \mathcal{C}$ and $\mathcal{D}$ in $\mathrm{Age}(\mathcal{M})$ satisfying:

1. $\mathcal{B}, \mathcal{C}' \leq \mathcal{D}$;
2. $\mathrm{cl}_{\mathcal{D}}(B) \cap C' \subseteq \mathrm{cl}_{\mathcal{D}}(A)$;
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Taking cl to be the identity recovers the usual notion of disjoint (strong) amalgamation.

Involving S_∞

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Equivalently,

1. for every finite $A, B, C \subseteq M$ with $C \subseteq A, B$, there is some $A' \cong_C A$ such that $A' \cap \mathrm{cl}(B) \subseteq \mathrm{cl}(C)$ and $\mathrm{cl}(A') \cap B \subseteq C$;

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3. same as (1) but assuming $|A \setminus C| = |B \setminus C| = 1$;
4. for every $C \subseteq M$ and $a, b \in M$, there is some $a' \cong_C a$ such that $a' \notin \mathrm{cl}(bC)$, $a' \notin \mathrm{cl}(aC)$ and $a \notin \mathrm{cl}(a'C)$.

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With equivalence (4) in mind, we can see the connection with Krk.

INVOLVING S_∞

ORDINAL RANK IMPLIES DISJOINT AMALGAMATION RELATIVE TO CL

- if we define $a \in \mathrm{cl}(\bar{c})$ iff $\mathrm{Krk}(a, \bar{c}) < \infty$, this is a disjointifying, aut-invariant, finitary closure operator.

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- Thus it is nontrivial iff $\mathrm{Krk}(a, \emptyset) = \infty$ for some a.

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- In fact it will be the minimal such closure operator on $\mathcal{M}$.
- Thus it is nontrivial iff $\mathrm{Krk}(a, \emptyset) = \infty$ for some a.
- One can now mimic the argument from Baldwin-Friedman-Koerwien-Laskowski to show $\mathrm{Aut}(\mathcal{M})$ involves S_∞.

Pinned property

Theory of pinned/unpinned equivalence relations developed by Hjorth and further explored by Larson and Zapletal.

Provides a very useful tool for proving negative results around Borel reducibility.

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Theorem (Hjorth)

Assuming AC, if G is cli then any orbit equivalence relation E_X^G is pinned. Furthermore, if $E \leq_B F$ and F is pinned, then E is pinned.

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Assuming AC, if G is cli then any orbit equivalence relation E_X^G *is pinned. Furthermore, if* $E \leq_B F$ *and F is pinned, then E is pinned.*

Conversely,

Corollary 1 (Hjorth, Gao, Thompson)

Assuming AC, if G is not cli, then there is an orbit equivalence relation E_X^G *which is not pinned.*

Pinned property

Definition 3.1

*Given equivalence relation E on Polish X, a **virtual** E**-class** is a pair $(\mathbb{P}, \tau)$ where $\mathbb{P}$ is a forcing poset and τ is a $\mathbb{P}$-name for an element of X such that*

$$\Vdash_{\mathbb{P}\times\mathbb{P}} \tau[\dot{G}_\ell] \; E \; \tau[\dot{G}_r].$$

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We write $(\mathbb{P}, \tau) \; \tilde{E} \; (\mathbb{Q}, \sigma)$ iff

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We say $(\mathbb{P}, \tau)$ is trivial iff there is some $x \in X$ such that

$$\Vdash_{\mathbb{P}} \tau[\dot{G}]\ E\ x.$$

Pinned property

It's straightforward to check $\tilde{E}$ defines an equivalence relation on the virtual E-classes.

We identify E-classes with the trivial virtual E-classes.

Some examples

1. For any subset $A \subseteq \mathbb{R}$, then $\mathbb{P} := \mathrm{coll}(A, \omega)$ and τ being the name for the added enumeration of A form a virtual $=^+$-class.

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2. For any ordinal α, then $\mathbb{P} := \mathrm{coll}(\alpha, \omega)$ and τ being a name for a LO on ω with ordertype α form a virtual E_{ω_1}-class.

Pinned property

Definition 3.2

*Say that E is **pinned** iff every virtual E-class is trivial.*

*A Polish group G has the **pinned property** iff every orbit equivalence relation E_X^G is pinned.*

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Theorem (Gao, Hjorth, Thompson)

Assuming AC, a Polish group has the pinned property iff it is cli.

Pinned property

Definition 3.3

The size of a virtual class $(\mathbb{P}, \tau)$ *is the least* κ *such that* $\mathrm{coll}(\omega, \kappa)$ *forces that* $(\mathbb{P}, \tau)$ *is trivial.*

An equivalence relation E is κ*-pinned iff every virtual E-class has size at most* κ*.*

Easy to check that if $E \leq_B F$ and F is κ-pinned then E is κ-pinned.

Some interesting results and open questions in the Larson-Zapletal book.

PINNED PROPERTY

Theorem (Larson, Zapletal)

In the Solovay model constructed from a measurable cardinal, if E is an analytic equivalence relation which is unpinned, at least one of the following holds:

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Corollary 2

If G is non-Archimedean, then it has the pinned property in the Solovay model constructed from a measurable cardinal iff it does not involve S_∞.

Thank you!