Does the axiom of Dependent Choices imply the axiom of Countable Choices, locally?

joint work with Lorenzo Notaro

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Some well-known principles

DC(X), the axiom of Dependent Choices for X

$\forall R \subseteq X^2\, (\forall x\, \exists y\, (x\ R\ y) \Rightarrow \exists (x_n)_{n\in\omega}\, \forall n\, (x_n\ R\ x_{n+1}))$.

(DC) $\forall X \neq \emptyset\, \mathsf{DC}(X)$

$\mathsf{AC}_\omega(X)$ i.e. CC(X), the axiom of Countable Choices for X

$\forall (A_n)_{n\in\omega} \big(\forall n\, (\emptyset \neq\ A_n \subseteq X) \Rightarrow \exists (x_n)_{n\in\omega}\, \forall n\, (x_n \in A_n)\big)$.

(AC_ω) $\forall X\, \mathsf{AC}_\omega(X)$

AC(X), the axiom of choice for X

$\exists F\colon \mathscr{P}(X) \to X\, \forall Y \subseteq X\, (Y \neq \emptyset \Rightarrow F(Y) \in Y)$.

(AC) $$\forall X \neq \emptyset\, \mathsf{AC}(X)$$

For all $X \neq \emptyset$, AC(X) $\Rightarrow$ DC(X) and AC(X) $\Rightarrow$ $\mathsf{AC}_\omega(X)$,
and DC $\Rightarrow$ AC_ω.

Question:

Does DC(X) $\Rightarrow$ $\mathsf{AC}_\omega(X)$ for all X?

To avoid trivialities we assume from now on that X is infinite, i.e. not in bijection with any $n \in \omega$. In particular $X \neq \emptyset$.

Easy facts

If Y is the surjective image of X, then $\mathsf{DC}(X) \Rightarrow \mathsf{DC}(Y)$.

If $Y \precsim X$, that is Y injects into X, then $\mathsf{AC}_\omega(X) \Rightarrow \mathsf{AC}_\omega(Y)$.

$\mathsf{DC}(X)$ is equivalent to the stronger form:
$\forall R \subseteq X^2\,(\forall x\,\exists y\,(x\ R\ y) \Rightarrow \forall \bar{x} \in X\,\exists (x_n)_{n\in\omega}\,(\bar{x} = x_0 \wedge \forall n\,(x_n\ R\ x_{n+1})))$.

If $\mathsf{DC}(X)$ and the $\emptyset \neq A_n \subseteq X$ are pairwise disjoint, then there is $(a_n)_n$ such that $a_n \in A_n$, for all $n \in \omega$.

Proof.

Require that every $x \in A_n$ is R-related to every $y \in A_{n+1}$. □

If the A_ns are not disjoint, the R-chain given by $\mathsf{DC}(X)$ might be a loop, meeting only finitely many A_ns.

$\mathsf{DC}(X \times \omega) \Rightarrow \mathsf{AC}_\omega(X)$.

Proof.

Given $A_n \subseteq X$, the sets $A_n \times \{n\} \subseteq X \times \omega$ are pairwise disjoint. □

If $X \times 2 \precsim X$, then $X \times \omega \precsim X$.

Proof.

If $f_0, f_1 \colon X \to X$ are injective and $\operatorname{ran} f_0 \cap \operatorname{ran} f_1 = \emptyset$, then $F \colon X \times \omega \to X$ defined by $F(x, n) = f_0 \circ \underbrace{f_1 \circ \cdots \circ f_1}_{n \text{ times}}(x)$ is injective □

Theorem 1a

If $X \times 2 \precsim X$, then $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$.

In particular $\mathsf{DC}(\mathbb{R}) \Rightarrow \mathsf{AC}_\omega(\mathbb{R})$. As every X is contained in some Y such that $Y \times 2 \precsim Y$, then $\mathsf{DC} \Rightarrow \mathsf{AC}_\omega$.

Theorem 1b

Assume $\mathsf{AC}_\omega(\mathbb{R})$. Then $\mathsf{DC}(A) \Rightarrow \mathsf{AC}_\omega(A)$ for all infinite sets A.

Proof.

Assume $\mathsf{DC}(A)$ and let $\emptyset \neq A_n \subseteq A$. Define $I \colon A \to \mathscr{P}(\omega)$, $I(a) = \{n \in \omega \mid a \in A_n\}$. Let $X_n = \{x \in \operatorname{ran} I \mid n \in x\} \subseteq \mathscr{P}(\omega)$. If $a \in A_n$ then $n \in I(a)$ so $I(a) \in X_n$ and hence $\emptyset \neq X_n$ for all $n \in \omega$. By $\mathsf{AC}_\omega(\mathbb{R})$ pick $x_n \in X_n$—the x_ns need not be distinct! Let $B_n = I^{-1}(\{x_n\}) \subseteq A$. Then $B_n = \{a \mid I(a) = x_n\} = \{a \mid \{k \mid a \in A_k\} = x_n\}$ and since $n \in x_n$, then $B_n \subseteq A_n$. The B_ns need not be distinct, but if $x_n \neq x_m$ then $B_n \cap B_m = \emptyset$, so after some trivial reindexing, by $\mathsf{DC}(A)$ we can pick $a_n \in B_n \subseteq A_n$. □

There is another, more constructive proof of Theorem 1b, proving that $\mathsf{DC}(A)$ implies $\mathsf{AC}_\omega^{\mathrm{fin}}(A)$ the axiom of countable choices for finite subsets of A.

Theorem 1

Assume one of the following:

- $\mathfrak{m} + \mathfrak{m} = \mathfrak{m}$ for all $\mathfrak{m}$ (i.e. $X \times 2 \precsim X$ for all infinite X)
- $\mathsf{AC}_\omega(\mathbb{R})$.

Then $\forall X\, (\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X))$.

Remarks

It is consistent with ZF that:

- $\mathsf{AC}_\omega(\mathbb{R})$ fails and $\mathfrak{m} + \mathfrak{m} = \mathfrak{m}$ for all $\mathfrak{m}$ (Sageev, 1975)
- $\mathsf{AC}_\omega(\mathbb{R})$ holds and there is an infinite X such that $X \times 2$ is not in bijection with X

Theorem 2

It is consistent with ZF that there is $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ holds, but $\mathsf{AC}_\omega(A)$ fails.

An equivalent (?) version of dependent choices

$\mathsf{DC}_\omega(X)$

Let $\emptyset \neq T \subseteq {}^{<\omega}X$ be a (descriptive-set-theoretic) tree on X. If T is pruned then it has an infinite branch.

$\mathsf{DC}_\omega(X) \Rightarrow \mathsf{DC}(X)$.

Proof.

Given $R \subseteq X \times X$ as in the definition of $\mathsf{DC}(X)$, the tree T of attempts to build an R-chain is pruned, and any branch of T is the desired sequence. □

$\mathsf{DC}({}^{<\omega}X) \Rightarrow \mathsf{DC}_\omega(X)$

Proof.

Given T on X, then ${}^{<\omega}X$ maps onto T so $\mathsf{DC}(T)$ holds. For $s, t \in T$ set $s \mathrel{R} t$ iff t is an immediate extension of s. □

Therefore

$$\mathsf{DC} \Leftrightarrow \mathsf{DC}_\omega.$$

$\mathsf{DC}_\omega(X) \Rightarrow \mathsf{AC}_\omega(X)$ for all infinite X. In particular $\mathsf{DC}(X)$ does not imply $\mathsf{DC}_\omega(X)$ in ZF.

Proof.

Given $\emptyset \neq A_n \subseteq X$, the tree $T = \{s \in {}^{<\omega}X \mid \forall i < \operatorname{lh} s\, (s(i) \in A_i)\}$ is pruned, and any branch of T yields a desired sequence. □

Let $X \subseteq \mathbb{R}$ be infinite and such that $\mathsf{DC}(X)$ holds.

1. For any $x \in \mathrm{Cl}\, X$ there is a sequence $x_n \in X$ converging to x. In particular X is Dedekind infinite.
2. If moreover
   - either$X$ contains a perfect set,
   - or $X \times 2 \precsim X$,

   then $\mathsf{AC}_\omega(X)$ holds.

Proof.

1 is easy.

2 If X contains a perfect set, then $\mathbb{R}$ embeds into X, hence X and $\mathbb{R}$ would be in bijection, so Therefore

$\mathsf{DC}(X) \Rightarrow \mathsf{DC}(\mathbb{R}) \Rightarrow \mathsf{AC}_\omega(\mathbb{R}) \Rightarrow \mathsf{AC}_\omega(X)$. □

Theorem 2

There is a model M of ZF in which there is $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ holds but A is non-separable.

The model M is a symmetric extension, that is $\mathrm{V} \subset M \subset \mathrm{V}[G]$. The forcing $\mathbf{P}$ adds an ω_1-sequence of reals $\langle x_\alpha \mid \alpha < \omega_1 \rangle$, and the set A is $\{x_\alpha \mid \alpha < \omega_1\} \in M$, but the enumeration $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ does not belong to M. The model M is obtained by taking the interpretation of hereditary symmetric names— the fact that A is non-separable, and hence $\mathsf{AC}_\omega(A)$ fails, as in Cohen's model.
The difficult bit is to guarantee $\mathsf{DC}(A)$.

The forcing

$\mathbf{P} = \bigcup_{n \in \omega} \mathbf{P}_n$ where $\mathbf{P}_0 \subseteq \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \ldots$.

These are just inclusions, not complete embeddings.
We write $\Vdash_n$, 1_n and $\leq_n$ instead $\Vdash_{\mathbf{P}_n}$, $1_{\mathbf{P}_n}$ and $\leq_{\mathbf{P}_n}$.

$\mathbf{P}_0 = \{\emptyset\}$ is the trivial forcing.

$\mathbf{P}_{n+1}$ is the set of all functions p, where $\operatorname{dom} p \subseteq \omega_1$ is countable and there is $X \subseteq \operatorname{dom} p$ such that:

- $p{\restriction}X \in \mathbf{P}_n$,
- $p(\alpha)$ is a $\mathbf{P}_n{\restriction}X$-name and $p{\restriction}X \Vdash_{\mathbf{P}_n} (p(\alpha)$ is in ${}^{<\omega}2)$, for $\alpha \notin X$,
- $p{\restriction}X \Vdash_n \forall s \exists t(s \subseteq t$ and $\forall x \in \dot{E}_{p,x}\,(t \nsubseteq x))$, where $\dot{E}_{p,x} = \{(p(\alpha), 1_n) \mid \alpha \in \operatorname{dom} p \setminus X\}$.

For each $p \in \mathbf{P}_{n+1}$ one can pick a largest X as above, call it $X(p)$.
$p \leq_{n+1} q$ iff

- $\operatorname{dom} p \supseteq \operatorname{dom} q$,
- $p{\restriction}X(p) \leq_n q{\restriction}X(q)$,
- if $\alpha \notin X(q)$ then $p{\restriction}X(p) \Vdash_n q(\alpha) \subseteq p(\alpha)$.

(There is one further technical requirement, but let's forget about it.)

$\mathbf{P}$ adds ω_1 new reals whose names are $\dot{a}_\alpha$ $(\alpha < \omega_1)$, and a new set with name $\dot{A} = \{(\dot{a}_\alpha, 1) \mid \alpha \in \omega_1\}$.

As $\mathrm{V} \vDash \mathsf{ZFC}$, so does $\mathrm{V}[G]$, and in order to construct a model of ZF where choice fails we must pass to an intermediate transitive class M.

M is obtained by choosing a suitable family of subgroups of the group of all permutations of $\mathbf{P}$, called a symmetric system. Using this we define the collection of hereditarily symmetric names HS, and we set

$$M = \{\dot{x}_G \mid x \in \mathsf{HS}\}.$$

Every canonical name for a set in V is in HS, so $\mathrm{V} \subseteq M$. Moreover $A \coloneqq \dot{A}_G \in M$ and $M \vDash A$ is non-separable.
The convoluted definition of $\mathbf{P}$ is needed to prove that $M \vDash \mathsf{DC}(A)$

The symmetric system

Any bijection $\pi \colon \omega_1 \to \omega_1$ yields an automorphism $\tilde{\pi}_n \colon \mathbf{P}_n \to \mathbf{P}_n$, $p \mapsto \tilde{\pi}_n p$:

- $\tilde{\pi}_0$ is the identity, since $\mathbf{P}_0$ is a singleton
- $\tilde{\pi}_{n+1} p\big(\pi(\alpha)\big) = \tilde{\pi}_n(p(\alpha))$.

The automorphism agree, i.e. $\tilde{\pi}_{n+1} \restriction \mathbf{P}_n = \tilde{\pi}_n$ so we get an automorphism $\tilde{\pi} \colon \mathbf{P} \to \mathbf{P}$.

Let $\mathcal{G}$ be the group of all automorphisms of $\mathbf{P}$ induced by a permutation $\pi \colon \omega_1 \to \omega_1$, and let $\mathcal{F} = \{\mathrm{fix}(E) \mid E \subseteq \omega_1 \text{ countable}\}$, where

$$\mathrm{fix}(E) = \{\tilde{\pi} \in \mathcal{G} \mid \forall \alpha \in E \, (\pi(\alpha) = \alpha)\}.$$

$\mathcal{F}$ is a filter of subgroups for $\mathcal{G}$, and $(\mathbf{P}, \mathcal{G}, \mathcal{F})$ is the symmetric system.

Thank You