Almost disjoint families in dimension 2 and higher

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Introduction

The goal of this talk is to give such an overview of the proof of:

Theorem (Schrittesser-Törnquist, 2021-22)

(ZF+DC+R-Unif) Let $\mathcal{I}$ be an iterated Frechet ideal. If all sets are (completely) Ramsey then there are no infinite $\mathcal{I}$-mad families.

Background

In this talk:

- $[\mathbb{N}]^\infty$ = infinite subsets of $\mathbb{N}$.
- $\mathsf{FIN} = [\mathbb{N}]^{<\infty}$ = finite subsets of $\mathbb{N}$.
- A family $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ is **almost disjoint** if for any **distinct** $x, y \in \mathcal{A}$, the intersection $x \cap y \in \mathsf{FIN}$.
- A **mad family** is a maximal almost disjoint family.

In recent years, there has been a lot new results about the (un-)definability of infinite mad families. One such theorem is the following:

Theorem (Schrittesser-Törnquist, 2018-2019)

(ZF+DC+R-Unif) If all sets are (completely) Ramsey then there are no infinite mad families.

This talk is about generalizing this to the situation where FIN is replaced by **Fubini products** of FIN with itself. *Especially the 2-dimensional case,*

$$\mathsf{FIN}^2 = \mathsf{FIN} \otimes \mathsf{FIN} = \{\mathbf{X} \subseteq \mathbb{N}^2 \mid \mathbf{X} \text{ has finitely many infinite verticals}\}.$$

Notation

You've already noticed I used a boldface capital letter for a subset $\mathbf{X} \subseteq \mathbb{N}^2$. *I'll stick to that throughout the talk.*

I also talked about **verticals**: If $\mathbf{X} \subseteq \mathbb{N}^2$ and $n \in \mathbb{N}$, the **vertical at** n is

$$\mathbf{X}(n) = \{m \in \mathbb{N} : (n, m) \in \mathbf{X}\}.$$

So: $\mathrm{FIN}^2 = \{\mathbf{X} \subseteq \mathbb{N}^2 : \{n \in \mathbb{N} : |\mathbf{X}(n)| = \infty\}$ is finite$\}$.

You also noticed I said *completely Ramsey*. Recall:

- For $a \subset \mathbb{N}$ finite and $A \subseteq \mathbb{N}$, $a \sqsubseteq A$ means a is an initial segment of A.
- When $a \sqsubseteq A$ and A is infinite,

  $$[a, A] = \{B \in [\mathbb{N}]^\infty : a \sqsubseteq B \subseteq A\} \qquad \text{(Ellentuck open nbhd)}$$

  Note: $[\emptyset, A] = [A]^\infty =$ all infinite subsets of A.
- $\mathcal{S} \subseteq [\mathbb{N}]^\infty$ is **completely Ramsey** if for every **finite** $a \subseteq \mathbb{N}$ and **infinite** $A \subseteq \mathbb{N}$ with $a \sqsubseteq A$ there is $B \in [a, A]$ such that

  $$[a, B] \subseteq \mathcal{S} \text{ or } [a, B] \cap \mathcal{S} = \emptyset.$$

The theorem

Definition

- Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is called $\mathcal{I}$-disjoint if for any **distinct** $x, y \in \mathcal{A}$ we have $x \cap y \in \mathcal{I}$.
- An $\mathcal{I}$-disjoint family is $\mathcal{I}$-mad (or m$\mathcal{I}$d) if it is a maximal $\mathcal{I}$-disjoint family.
- An **iterated Frechet ideal** is an ideal that arises as a finite or infinite iteration of the Fubini product operation $\otimes$, starting from FIN.

E.g., $\mathrm{FIN} \otimes \mathrm{FIN} = \{\mathbf{X} \subseteq \mathbb{N}^2 \mid \{n \in \mathbb{N} : \mathbf{X}(n) \notin \mathrm{FIN}\} \in \mathrm{FIN}\} = \mathrm{FIN}^2$.

Theorem (Schrittesser-Törnquist, 2021-22)

(ZF+DC+R-Unif) Let $\mathcal{I}$ be an iterated Frechet ideal. If all sets are (completely) Ramsey then there are no infinite $\mathcal{I}$-mad families.

In particular: *If all sets are Ramsey then there are no infinite* FIN^2*-mad families.*

What is *really* in this talk

I this talk, I will sketch a proof of the following special case of the theorem:

Theorem (Haga-Schrittesser-T., 2016)

There are no infinite analytic FIN^2*-mad families.*

Remarks:

- The proof I will sketch is based on the Ramsey-theoretic approach developed for the much more general theorem on the previous slide.
- The original proof due to Haga-Schrittesser-T. used forcing and absoluteness (and proved much more beyond analytic sets.)
- In the analytic case, we don't to say anything about R-Unif, since in this case we have the Jankov-von Neumann uniformization theorem, and we don't need to say anything about completely Ramsey, because all analytic sets are completely Ramsey.

For the remainder of the talk, we fix $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ **which is an infinite analytic** FIN^2**-almost disjoint family.** *We will eventually prove that* $\mathcal{A}$ *is not maximal.*

From 1 to 2 dimensions: The tilde operator

The key idea behind the Ramsey-theoretic proof that there are no infinite analytic FIN^2-mad families is an operator, which we call the **tilde operator**, which transforms an infinite set $A \subseteq \mathbb{N}$ to a set $\tilde{A} \in \mathrm{FIN}^{2+}$.

Let $A \subseteq \mathbb{N}$ be infinite. Recall: $\mathcal{A}$ is our fixed, infinite FIN^2-a.d. family.

As a small simplifying assumptions, we suppose we have a sequence $\mathbf{Z}^\ell \in \mathcal{A}$, where $\ell \in \mathbb{N}$, such that

- all non-empty columns (i.e., verticals) of $\mathbf{Z}^\ell$ are infinite;
- $\ell \neq m \implies \mathbf{Z}^\ell \cap \mathbf{Z}^m = \emptyset$.

Denote by $\hat{\mathbf{Z}}^\ell(m, n)$ the entry $(p, q) \in \mathbf{Z}^\ell$ where p is the first coordinate of the m'th non-$\emptyset$ column of $\mathbf{Z}^\ell$, and q is the n'th entry in this column.

Definition (Definition of $\tilde{A}$, given $A \subseteq \mathbb{N}$)

$\tilde{A} = \{\hat{\mathbf{Z}}^\ell(m, n) : \ell, m, n \in A \wedge \ell < m < n \wedge \ell$ and m are consecutive in $A\}$.

Note: It is easy to show that $\tilde{A} \in \mathrm{FIN}^{2+}$.

Facts about the tilde operator

Theorem (Genericity theorem for the tilde operator; Schrittesser-T.)

The set

$$\{A \in [\mathbb{N}]^\infty : \tilde{A} \textit{ is } \mathrm{FIN}^2\textit{-almost disjoint from every } \mathbf{X} \in \mathcal{A}\}$$

is Ramsey co-null, and so $\mathcal{A}$ is **not** *maximal.*

Important facts about the tilde operator

Below, a, X, A, A' denote subsets of $\mathbb{N}$ (a finite, A, A' infinite by default).

1. $(\mathrm{FIN}, \mathrm{FIN}^2)$-**equivariance:** If $A \triangle A' \in \mathrm{FIN}$, then $\tilde{A} \triangle \tilde{A}' \in \mathrm{FIN}^2$.
2. **Pigeon hole principle for the domain**: For any $[a, A]$ there is $B \in [a, A]$ such that either $\mathrm{dom}(\tilde{B}) \cap X \in \mathrm{FIN}$ or $\mathrm{dom}(\tilde{B}) \subseteq^{\mathrm{FIN}} X$.
3. **Pigeon hole principle for the verticals:** For any $[a, A]$ and $m \in \mathrm{dom}(\tilde{A})$, there is $B \in [a, A]$ such that $m \in \mathrm{dom}(\tilde{B})$, and either $\tilde{B}(m) \cap X \in \mathrm{FIN}$ or $\tilde{B}(m) \subseteq^{\mathrm{FIN}} X$.
4. **Almost disjointness principle:** For any $\mathbf{A} \in \mathcal{A}$ and $[a, A]$, there is $B \in [a, A]$ such that $\tilde{B} \cap \mathbf{A} \in \mathrm{FIN}^2$.

Tree representations

To prove the the genericity theorem for the tilde operator, we need to work with the tree representations of analytic sets in $\mathcal{P}(\mathbb{N} \times \mathbb{N})$.

From now on, we'll usually identify $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ with $2^{\mathbb{N} \times \mathbb{N}}$.

Tree representations for analytic subsets of $2^{\mathbb{N} \times \mathbb{N}}$.

- For any analytic $S \subseteq 2^{\mathbb{N} \times \mathbb{N}}$, there is a **closed** set $F \subseteq 2^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $S = \pi(F)$.

  (Here $\pi : 2^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N} \times \mathbb{N}}$ is the projection.)
- From this F, we can obtain a "tree"

$$T = \{(s,t) \in \bigcup_{n \in \mathbb{N}} 2^{n \times n} \times \mathbb{N}^n : (\exists x, y \in F)\ x \supseteq s \wedge y \supseteq t\}.$$

- Then F is *exactly* the set of infinite branches through T, i.e. $F = [T]$.
- So $S = \pi[T]$.

Fix a tree T such that $p[T] = \mathcal{A} =$ our fixed FIN^2-disj. family.

The associated trees $T^{\mathbf{X}}$ and $T^{\mathbf{X},d}$

Recall: We have fixed an infinite analytic FIN^2-a.d. family $\mathcal{A}$, and a tree T such that $p[T] = \mathcal{A}$. **We'll write T_t for the subtree of those things in T that are compatible with t, i.e.,** $T_t = \{s \in T : s \subseteq t \vee t \subseteq s\}$.

Definition of the trees $T^{\mathbf{X}}$ and $T^{\mathbf{X},d}$

For each $\mathbf{X} \subseteq \mathbb{N}^2$ and $d \in \mathrm{FIN}$, define

$$T^{\mathbf{X}} = \{t \in T : (\exists \mathbf{A} \in p[T_t])\ \mathbf{A} \cap \mathbf{X} \in \mathrm{FIN}^{2+}\}.$$

$$T^{\mathbf{X},d} = \{t \in T : (\exists \mathbf{A} \in p[T_t])\ \mathbf{A} \cap \mathbf{X} \in \mathrm{FIN}^{2+} \wedge$$
for all $i \in d$, the intersection of the verticals $\mathbf{A}(i) \cap \mathbf{X}(i)$ are infinite$\}$.

Remark:

- Clearly $T^{\mathbf{X},\emptyset} = T^{\mathbf{X}}$.
- We think of $T^{\mathbf{X},d}$ as the tree of attempts to find an $\mathbf{A} \in \mathcal{A}$ such that $\mathbf{A} \cap \mathbf{X} \in \mathrm{FIN}^{2+}$ and such that for all $i \in d$, the intersection of the verticals $\mathbf{A}(i)$ and $\mathbf{X}(i)$, which sit above i, are infinite.

Invariance properties of $T^{\mathbf{X}}$ and $T^{\mathbf{X},d}$

Let me repeat the definition from the previous slide:

Definition of the trees $T^{\mathbf{X}}$ and $T^{\mathbf{X},d}$

For each $\mathbf{X} \subseteq \mathbb{N}^2$ and $d \in \mathrm{FIN}$, define

$$T^{\mathbf{X}} = \{t \in T : (\exists \mathbf{A} \in \mathcal{A} \in p[T_t])\ \mathbf{A} \cap \mathbf{X} \in \mathrm{FIN}^{2+}\}.$$

$$T^{\mathbf{X},d} = \{t \in T : (\exists \mathbf{A} \in \mathcal{A} \in p[T_t])\ \mathbf{A} \cap \mathbf{X} \in \mathrm{FIN}^{2+} \wedge$$

for all $i \in d$, the intersection of the verticals $\mathbf{A}(i) \cap \mathbf{X}(i)$ are infinite$\}$.

Lemma

1. **(Invariance)** *If* $\mathbf{X} \triangle \mathbf{X}' \in \mathrm{FIN}^2$, *then* $T^{\mathbf{X}} = T^{\mathbf{X}'}$.
2. **(Conditional invariance)** *If* $\mathbf{X} \triangle \mathbf{X}' \in \mathrm{FIN}^2$ *and for each* $i \in d$ *we have* $\mathbf{X}(i) \triangle \mathbf{X}'(i) \in \mathrm{FIN}$, *then* $T^{\mathbf{X},d} = T^{\mathbf{X}',d}$.

Proof: Clear by the definition of $T^{\mathbf{X}}$ and $T^{\mathbf{X},d}$. □

Stabilizing the functions $\mathbf{X} \mapsto T^{\mathbf{X},d}$ on a large set

Lemma

For any $A \in [\mathbb{N}]^\infty$ there is an infinite $B \subseteq A$ such that for every $X \in [B]^\infty$ and every finite $d \subseteq \mathrm{dom}(\tilde{X})$ we have $T^{\tilde{X},d} = T^{\tilde{B},d}$.

This means: *Inside any infinite set $A \subseteq \mathbb{N}$, we can find an infinite $B \subseteq A$ such that the functions $X \mapsto T^{\tilde{X},d}$ are "as constant as possible" on $[B]^\infty$.*

Proof: Since $X \mapsto T^{\tilde{X},d}$ is Baire measurable with respect to the Ellentuck topology on $[\mathbb{N}]^\infty$, we can go through the countably many pairs $(t, d) \in T \times \mathrm{FIN}$ and accept or reject the statement

$$\text{“ } t \in T^{\tilde{X},d} \text{ ”}.$$

□

For the rest of the talk, fix $A \subseteq \mathbb{N}$ such that the Lemma above holds. That is, we'll assume that $T^{\tilde{X},d} = T^{\tilde{A},d}$ for all $X \in [A]^\infty$.

The branch lemma (Proof in Appendix)

Lemma (Branch Lemma)

If $t_0, t_1 \in T^{\tilde{A},d}$ differ in the first component, then there is $d' \in [A]^{<\infty}$ with $\min d' > \max d$ and $t_0', t_1' \in T^{\tilde{A},d\cup d'}$ such that one of the following hold:

1. *For all $m > \max d'$ and all $(w_0, w_1) \in [T^{\tilde{A},d\cup d'}_{t_0'}] \times [T^{\tilde{A},d\cup d'}_{t_1'}]$*

$$m \notin \mathrm{dom}_\infty(\pi(w_0)) \cap \mathrm{dom}_\infty(\pi(w_1)).$$

   That is: *When we look below t_0' and t_1' in $T^{\tilde{A},d\cup d'}$, we won't find any w_0, w_1 with $\pi(w_0) \cap \pi(w_1)$ having a further infinite column.*

2. *There is $m \in d \cup d'$ and $n \in \mathbb{N}$ such that for all $w_0 \in [T^{\tilde{A},d\cup d'}_{t_0'}]$ and $w_1 \in [T^{\tilde{A},d\cup d'}_{t_1'}]$ we have:*

$$(\pi(w_0) \cap \pi(w_1))(m) \subseteq n.$$

   That is: *There is some $m \in d \cup d'$ so that if we look below t_0' and t_1' in $T^{\tilde{A},d\cup d'}$, we will never find w_0, w_1 where the intersection $\pi(w_0)(m) \cap \pi(w_1)(m)$ of the columns above m has grown out of n.*

Putting it all together: Claim 1

Recall that we're trying to prove:

Theorem (Genericity theorem for the tilde operator; Schrittesser-T.)

The set below is Ramsey co-null, and so $\mathcal{A}$ is **not** *maximal.*

$$\{A \in [\mathbb{N}]^\infty : \tilde{A} \text{ is } \mathrm{FIN}^2\text{-almost disjoint from every } \mathbf{X} \in \mathcal{A}\}$$

To finish the proof, we'll show that we must have $T^{\tilde{A}} = \emptyset$.

This is enough, since $T^{\tilde{A}}$ is the tree that searches for some $\mathbf{A} \in \mathcal{A}$ which intersects $\tilde{A}$ in a FIN^{2+} set, so if $T^{\tilde{A}} = \emptyset$ then $\tilde{A} \cap \mathbf{A} \in \mathrm{FIN}^2$ for all $\mathbf{A} \in \mathcal{A}$.

Claim 1: If $|p[T^{\tilde{A}}]| \leq 1$, then $T^{\tilde{A}} = \emptyset$.

Proof of Claim 1:

- Recall the **almost disjointness principle** for the tilde operator: For any $\mathbf{A} \in \mathcal{A}$ and $[a, A]$, there is $B \in [a, A]$ such that $\tilde{B} \cap \mathbf{A} \in \mathrm{FIN}^2$.
- By this principle, we can find $B \in [A]^\infty$ with $\tilde{B} \cap \mathbf{A} \in \mathrm{FIN}^2$.
- But then $T^{\tilde{B}} = \emptyset$, whence $T^{\tilde{A}} = \emptyset$. $\square_{\text{Claim1}}$

Putting it all together: Claim 2

Claim 2: It is not possible to have $|p[T^{\tilde{A}}]| > 1$.

Before proving this claim (on the next slide), recall the pigeon hole principles for the tilde operator:

Important facts about the tilde operator

Let $X \subseteq \mathbb{N}$.

1. **Pigeon hole principle for the domain**: For any $[c, C]$ there is $B \in [c, C]$ such that either $\operatorname{dom} \tilde{B} \cap X \in \mathrm{FIN}$ or $\operatorname{dom} \tilde{B} \subseteq^{\mathrm{FIN}} X$.
2. **Pigeon hole principle for the verticals:** For any $[c, C]$ and $m \in \operatorname{dom}(\tilde{C})$, there is $B \in [c, C]$ such that $m \in \operatorname{dom} B$, and either $\tilde{B}(m) \cap X \in \mathrm{FIN}$ or $\tilde{B}(m) \subseteq^{\mathrm{FIN}} X$.

Putting it all together: Claim 2 continued

Recall: **Claim 2:** It is not possible to have $|p[T^{\tilde{A}}]| > 1$.

Proof of Claim 2:

- If $p[T^{\tilde{A}}]$ has at least 2 elements, then we can find $t_0, t_1 \in T^{\tilde{A}}$ which are incompatible in the first coordinate.
- Then the **Branch Lemma** tells us that we can find $d \subseteq A$ finite and $t_0', t_1' \in T^{\tilde{A},d}$ with $t_0' \supseteq t_0$ and $t_1' \supseteq t_1$ such that for any $\mathbf{X}_0 \in p[T^{\tilde{A},d}_{t_0'}]$ and $\mathbf{X}_1 \in p[T^{\tilde{A},d}_{t_1'}]$ we have at least one of the following:
  1. If $i \notin d$, then $\mathbf{X}_0(i)$ and $\mathbf{X}_1(i)$ can't both be infinite.
  2. **or** a column above some $i \in d$ where $\mathbf{X}_0(i) \cap \mathbf{X}_1(i) \subseteq n$ (some fixed $n \in \mathbb{N}$).
- In either case, the **pigeon hole principles** for the tilde operator ensures that we can find $B \in [A]^\infty$ with $d \in \operatorname{dom}(\tilde{B})$, such that $t_0' \notin T^{\tilde{B},d}$, contradicting that $T^{\tilde{B},d} = T^{\tilde{A},d}$. $\square_{\text{Claim2}}$

$\square$

Thank for listening!

Appendix: Proof of the branch lemma

Proof of the Branch Lemma:

Claim: If the branch lemma fails for $t_0, t_1 \in T^d$, then there is $k > \max d$ and $t_0', t_1' \in T^{d \cup \{k\}}$, with $t_0' \supset t_0$ and $t_1' \supset t_1$, such that the Branch Lemma fails for t_0', t_1' and $d \cup \{k\}$.

Proof of Claim: If the branch lemma holds for t_0', t_1' and $d \cup \{k\}$, then it also holds for t_0, t_1 and d. $\square_{Claim}$

- Note now that if the branch lemma fails for t_0, t_1 and d, then for any given n we can take $t_0', t_1' \in T^{d \cup \{k\}}$ in the claim so that

$$\pi(t_0')(m) \cap \pi(t_1')(m) \not\subseteq n \text{ for all } m \in d \cup \{k\}.$$

- Using this, we can build a sequence

$$\max d < k_1 < k_2 < \dots$$

and infinitely growing $t_0^\ell, t_1^\ell \in T^{d \bigcup \{k_1, \dots, k_\ell\}}$ extending each other, which will build a branch in T where the 1st coordinates will have FIN^{2+} intersection, despite $\pi(t_0) \neq \pi(t_1)$. $\square$