Homeomorphism groups of Knaster continua

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Introduction

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Theorem (Ellis, 1960)

Let G be a topological group. There is a universal minimal flow for G and it is unique (up to G-flow isomorphism).

Denote the universal minimal flow of G by $\mathcal{M}(G)$.

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Examples of extremely amenable groups arise via:

(1) automorphism groups of Fraissé structures (closed subgroups of S_∞)–$\mathrm{Aut}(\mathbb{Q}, \leq)$ (Pestov)

(2) infinite-dimensional groups from analysis and measure theory– $U(\ell^2)$ (Gromov-Milman), $\mathrm{Aut}(X, \mu)$ (Giordano-Pestov)

Motivation: to study universal minimal flows of homeomorphism groups of indecomposable continua–the pseudo-arc and Knaster continua.

A **continuum** is a compact, connected, metric space

A continuum X is **indecomposable** if $X = A \cup B$ for subcontinua A, B implies $A = X$ or $B = X$.

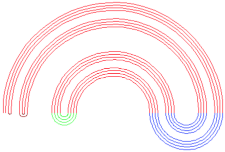

Figure: Knaster's buckethandle continuum

Question: (Uspenskij, 2000) What is the universal minimal flow of $\mathrm{Homeo}(P)$, for P the pseudo-arc? (pseudo-arc = chainable and hereditarily indecomposable)

Knaster continua = simpler than the pseudo-arc but still indecomposable

Examples

$\mathrm{Homeo}_+[0,1]$ — extremely amenable (Pestov)

$\mathcal{M}(\mathrm{Homeo}_+(\mathbb{S}^1)) \simeq \mathbb{S}^1$ (Pestov)

$\mathcal{M}(\mathrm{Homeo}(L))$, for L the Lelek fan is metrizable (Bartošová and Kwiatkowska)

$\mathcal{M}(\mathrm{Homeo}(P)) =??$

Knaster continua and main theorem

Knaster continua

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Ex. buckethandle $= \varprojlim (I_n, s_2)$ where

$$s_2(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2 - 2x & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

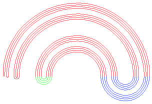

Figure: Knaster's buckethandle continuum

Main theorem

Define: the **universal Knaster continuum** to be a Knaster continuum which continuously and openly surjects onto all other Knaster continua

K will be the universal Knaster continuum

Theorem (I., 2022)

The group $Homeo(K)$ is isomorphic as a topological group to

$$U \rtimes F$$

where U is a Polish extremely amenable group and F is the free abelian group on countably many generators.

Corollary

$\mathcal{M}(Homeo(K))$ *is homeomorphic to* $\mathcal{M}(F)$.

Projective Fraïssé limits

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A **projective Fraissé category**, $\mathcal{F}$, is a countable (up to isomorphism) category of finite graphs and morphisms

1. each morphism is an epimorphism
2. $\mathcal{F}$ satisfies the joint projection property
3. $\mathcal{F}$ satisfies the **projective amalgamation property**.

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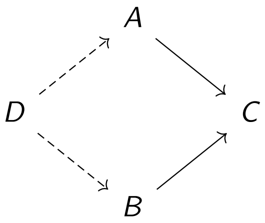
Amalgamation property:



A **topological graph** is a graph such that the underlying set X is a compact, metrizable, zero-dimensional space and R^X is closed.
(e.g., each finite graph with discrete topology)

Definition: for $\mathcal{F}$ a Fraissé class, let $\mathcal{F}^\omega$ be all topological graphs formed as inverse limits of a sequence of structures in $\mathcal{F}$ via morphisms in $\mathcal{F}$.

Theorem (Irwin, Solecki, 2006)

Let $\mathcal{F}$ be a projective Fraissé class. There exists a unique (up to isomorphism) topological graph $\mathbb{F} \in \mathcal{F}^\omega$ so that:

1. *for each $A \in \mathcal{F}$, there is a morphism $\mathbb{F} \to A$*
2. *for $A, B \in \mathcal{F}$, morphisms $f : \mathbb{F} \to A$ and $g : B \to A$, there is a morphism $h : \mathbb{F} \to B$ with $f = g \circ h$.*

Morphisms in $\mathcal{F}^\omega$:

Ramsey categories and the KPT correspondence

Ramsey categories

Let $\mathcal{F}$ be a projective Fraisse category.

Notation: for $A, B \in \mathcal{F}$, $\mathrm{hom}(B, A)$ is the set of all morphisms $B \to A$.

We say $A \in \mathcal{F}$ has the **Ramsey property** if for every $B \in \mathcal{F}$ and $d \in \mathbb{N}$ there exists $C \in \mathcal{F}$ so that for any coloring $c : \mathrm{hom}(C, A) \to d$, there exists some $f \in \mathrm{hom}(C, B)$ such that

$$\mathrm{hom}(B, A) \circ f \text{ is } c\text{-monochromatic.}$$

$\mathcal{F}$ is a **Ramsey category** if every object in $\mathcal{F}$ has the Ramsey property.

Theorem (Kechris, Pestov, Todorcevic, 2005)

Let $\mathcal{F}$ be a projective Fraïssé category with limit $\mathbb{F}$. The following are equivalent:

1. $Aut(\mathbb{F})$ *is extremely amenable.*
2. $\mathcal{F}$ *is a Ramsey category and members of* $\mathcal{F}$ *are rigid.*

The **Ramsey degree** of $A \in \mathcal{F}$ is the minimum $m \in \mathbb{N}$ such that for any $n > m$ and $B \in \mathcal{F}$, there is $C \in \mathcal{F}$ so that for any coloring $c : \hom(C, A) \to n$, there exists $f \in \hom(C, B)$ such that

$$|\hom(B, A) \circ f| \leq m$$

Theorem (Zucker, 2016)

Let $\mathcal{F}$, $\mathbb{F}$ be as before. Then, $\mathcal{M}(Aut(\mathbb{F}))$ is metrizable if and only if every member of $\mathcal{F}$ has finite Ramsey degree and members of $\mathcal{F}$ are rigid.

Ideal situation

The ideal is: start with a projective Fraïssé class $\mathcal{C}$ such that :

1. $R^{\mathbb{C}}$ is *transitive*
2. $\mathbb{C}/R^{\mathbb{C}}$ is homeomorphic to the continua C you care about
3. the category $\mathcal{C}$ "approximates" homeomorphisms of C well; i.e, $\mathrm{Aut}(\mathbb{C})$ is dense in $\mathrm{Homeo}(C)$

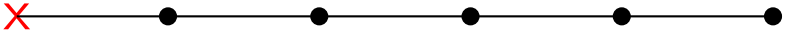
Then, compute the universal minimal flow of $\mathrm{Aut}(\mathbb{C})$ via KPT

Fact: If $H \leq G$ is dense and H is extremely amenable, then so is G.

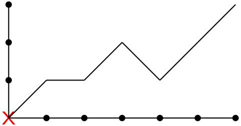
Approximating the universal Knaster continuum

Projective Fraïssé construction

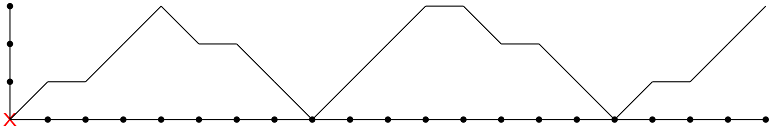
Let $\mathcal{K}$ be the category of all finite, reflexive, connected linear graphs with a marked endpoint.



epimorphisms = surjective maps that preserve R and marked endpoint



Morphisms in $\mathcal{K}$ are "tent-like" maps:



Degree:

Proposition

$\mathcal{K}$ is a projective Fraïssé category.

Let $\mathbb{K}$ be the Fraïssé limit of $\mathcal{K}$. Then:

Theorem

The relation $R^{\mathbb{K}}$ is a closed equivalence relation and

1. *$\mathbb{K}/R^{\mathbb{K}}$ is homeomorphic to the universal Knaster continuum*
2. *$Aut(\mathbb{K})$ is a dense subgroup of $Homeo(K)$*

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Theorem

The relation $R^{\mathbb{K}}$ is a closed equivalence relation and

1. $\mathbb{K}/R^{\mathbb{K}}$ *is homeomorphic to the universal Knaster continuum*
2. $Aut(\mathbb{K})$ *is a dense subgroup of* $Homeo(K)$

It turns out: $\mathcal{K}$ contains an object with infinite Ramsey degree
$\implies \mathcal{M}(\mathrm{Aut}(\mathbb{K}))$ is non-metrizable.

Note: this gives no information about $\mathrm{Homeo}(K)$...

Solution: consider a modified class, $\mathcal{K}^*$.

Objects: pairs (A, q) where A is a finite pointed linear graph and $q \in \mathbb{Q}^{>0}$

Morphisms: $f : (B, r) \to (A, q)$ is a morphism if $f : B \to A$ is a morphism in $\mathcal{K}$ and $\deg(f) = \frac{r}{q}$

Proposition

The category $\mathcal{K}^*$ is a Ramsey category.

The proof uses the classical Ramsey theorem.

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So: $\mathrm{Aut}(\mathbb{K}^*)$ is extremely amenable (by KPT)

What does this mean for $\mathrm{Homeo}(K)$?

Turns out: $\mathrm{Aut}(\mathbb{K}^*)$ is dense in the subgroup $\mathrm{Homeo}^1(K)$ of **degree one homeomorphisms** of K

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Debski, 1985– defines a notion of a **degree** for continuous open maps between Knaster continua

Theorem (Debski, 1985)

There is a continuous map $\deg : Homeo(K) \to \mathbb{Q}^\times$ *which is a group homomorphism.*

(Here, $\mathbb{Q}^\times$ is the group of positive rationals with multiplication.)

Main theorem again

Theorem

The group $Homeo(K)$ is isomorphic as a topological group to $U \rtimes F$ where U is Polish and extremely amenable and F is the free abelian group on countably many generators.

The U above is exactly $\mathrm{Homeo}^1(K)$.

So: $\mathcal{M}(\mathrm{Homeo}(K))$ is homeomorphic to $\mathcal{M}(\mathbb{Q}^\times)$ and the action of $\mathrm{Homeo}(K)$ on $\mathcal{M}(\mathbb{Q}^\times)$ is exactly the action via the degree map.

Thank you for listening :)