Polish modules over subrings of $\mathbb{Q}$

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This is joint work with Sławomir Solecki.

Introduction

Our aim

Give a construction of Polish modules.

Using the construction, we answer a question of Frisch and Shinko.

A corollary of our result that does not require more definitions:

There is a family of size $2^{\aleph_0}$ *of* F_σ $\mathbb{Q}$*-vector subspaces of* $\mathbb{R}$ *such that, for any* $F_1 \neq F_2$ *in the family, each Borel* $\mathbb{Q}$*-linear map* $F_1 \to F_2$ *is constantly equal to* 0*.*

Polish modules

R a countable commutative ring

Polish R-Modules = abelian Polish groups with a continuous scalar multiplication (with R equipped with the discrete topology)

$M_1 \sqsubseteq^R M_2$ if there is an injective continuous R-module homomorphism

$M_1 \sqsubset^R M_2$ if $M_1 \sqsubseteq^R M_2$ and $M_2 \not\sqsubseteq^R M_1$

Minimal Polish $\mathbb{Q}$-modules

Theorem (Frisch–Shinko)

For R a countable Noetherian ring, there is a countable family N_n, $n \in \mathbb{N}$, of uncountable Polish R-modules, such that, for each uncountable Polish R-module M,

$$N_n \sqsubseteq^R M \text{ for some } n.$$

If R is a countable field, then there is a **single** *uncountable Polish R-vector space N_R such that, for each uncountable Polish R-vector space M,*

$$N_R \sqsubseteq^R M.$$

$$N_{\mathbb{Q}} \sqsubseteq^{\mathbb{Q}} \mathbb{R}$$

$$N_{\mathbb{Q}} \sqsubset^{\mathbb{Q}} \mathbb{R}$$

Question(Frisch–Shinko): Is there anything in between?

Main theorem

Theorem (H.–Solecki)

R a subring of $\mathbb{Q}$ not equal to $\mathbb{Z}$

There exists V_x, $x \subseteq \mathbb{N}$, uncountable Polish R-modules such that

(i) $V_x \sqsubseteq^R \mathbb{R}$;

(ii) *if $x \setminus y$ is finite, then* $V_x \sqsubseteq^R V_y$;

(iii) *if $x \setminus y$ is infinite, then each continuous R-module homomorphism $V_x \to V_y$ is identically equal to zero; in particular,* $V_x \not\sqsubseteq^R V_y$.

Why theorem answers the question

Take $R = \mathbb{Q}$. Take x such that both x and $\mathbb{N} \setminus x$ are infinite. We have

$$N_{\mathbb{Q}} \sqsubset^{\mathbb{Q}} V_x \sqsubset^{\mathbb{Q}} \mathbb{R}.$$

Let $y = \mathbb{N} \setminus x$.

$N_{\mathbb{Q}} \sqsubseteq^{\mathbb{Q}} V_x, V_y \sqsubseteq^{\mathbb{Q}} \mathbb{R}$.

If $\mathbb{R} \sqsubseteq^{\mathbb{Q}} V_x$, then $V_y \sqsubseteq^{\mathbb{Q}} V_x$, contradiction.

If $V_x \sqsubseteq^{\mathbb{Q}} N_{\mathbb{Q}}$, then $V_x \sqsubseteq^{\mathbb{Q}} V_y$, again a contradiction.

Construction

Relevant objects

Base sequences;

Translation invariant analytic P-ideals;

Coherence condition between them.

Base sequences

Base sequence $\overrightarrow{a}$ = sequence (a_n) with $a_n \in \mathbb{N}$ and $a_n \geq 2$, for each $n \in \mathbb{N}$

A unique representation for $r \in \mathbb{R}_{\geq 0}$:

$$r = [r] + \sum_{n=1}^{\infty} \frac{r_n}{a_1 \cdots a_n},$$

where $[r]$ is the integer part of r, $0 \leq r_n < a_n$ and $r_n \neq a_n - 1$ for infinitely many n.

For a set P of primes define

$$P^{-1}\mathbb{Z} = \{\frac{k}{l} \mid k \in \mathbb{Z}, l \in \mathbb{N}, \text{ and, for each prime } p, \text{ if } p \mid l, \text{ then } p \in P\}.$$

Each subring of $\mathbb{Q}$ is of this form.

Each such subring is noetherian.

$\mathrm{pr}(\overrightarrow{a}) = \{p \mid \ p \text{ a prime and } p \mid a_n \text{ for all but finitely many } n\}$

$$\mathbb{Q}_{\overrightarrow{a}} = \left(\mathrm{pr}(\overrightarrow{a})^{-1}\right)\mathbb{Z}$$

Each subring of $\mathbb{Q}$ is of this form.

Ideals

Ideal = a non-empty family I of subsets of $\mathbb{N}$ that is closed under $\subseteq$ and finite $\cup$

I is **translation invariant** if for each $x \in I$,

$$\{n+1 \mid n \in x\} \in I \text{ and } \{n \mid n+1 \in x\} \in I.$$

I is an **analytic P-ideal** if I is analytic as a subsets of $2^{\mathbb{N}}$, and $\forall (x_n)$ sequence of elements of I, $\exists y \in I$ such that

$$x_n \setminus y \text{ is finite for each } n.$$

Equivalently, by Solecki's theorem, there is a lsc submeasure $\phi \colon 2^{\mathbb{N}} \to [0, \infty]$ such that

$$I = \mathrm{Exh}(\phi) = \{x \subseteq \mathbb{N} \mid \phi(x \setminus \{1, \ldots, n\}) \to 0, \text{ as } n \to \infty\}.$$

Coherence of base sequences and ideals

$\overrightarrow{a}$ is **adapted to** I if $\{n \mid a_n \neq a_{n+1}\} \in I$.

Constructing H

For non-negative r, let

$$j_{\vec{a}}(r) = \{n \in \mathbb{N} \mid r_n \neq r_{n+1}\}.$$

$j_{\vec{a}}$ collects “jumps” of digits.

$$H = \{r \mid r \in \mathbb{R}_{\geq 0} \text{ and } j_{\overrightarrow{a}}(r) \in I\} \cup \{-r \mid r \in \mathbb{R}_{\geq 0} \text{ and } j_{\overrightarrow{a}}(r) \in I\}$$

Lemma

If I is translation invariant and $\overrightarrow{a}$ is adapted to I, then

(i) *H is a subgroup of $\mathbb{R}$ taken with addition $+$.*

(ii) *H is closed under the multiplication by elements of $\mathbb{Q}_{\overrightarrow{a}}$.*

Topologizing H

I is an analytic P-ideal. Fix ϕ such that $I = \mathrm{Exh}(\phi)$. Define

$$\rho(r, s) = \phi\big(j(|r - s|)\big) + |r - s|, \text{ for } r, s \in \mathbb{R}.$$

ρ is almost a metric.

Lemma

There exists a metric d on H such that for each $\epsilon > 0$, there exists $\delta > 0$ with

$$\big(\rho(r, s) < \delta \Rightarrow d(r, s) < \epsilon\big) \text{ and } \big(d(r, s) < \delta \Rightarrow \rho(r, s) < \epsilon\big),$$

for all $r, s \in H$.

We call the topology induced by the metric d the **submeasure topology**.

Define

$I[\overrightarrow{a}] := H$ with the submeasure topology with addition $+$ and $\mathbb{Q}_{\overrightarrow{a}}$-scalar multiplication.

Polish $\mathbb{Q}_{\overrightarrow{a}}$-modules

Theorem

I a translation invariant, analytic P-ideal of subsets of $\mathbb{N}$

$\overrightarrow{a}$ a base sequence adapted to I

Then $I[\overrightarrow{a}]$ is a Polish $\mathbb{Q}_{\overrightarrow{a}}$-module and the identity map $I[\overrightarrow{a}] \to \mathbb{R}$ is a continuous $\mathbb{Q}_{\overrightarrow{a}}$-embedding.

Inclusions, non-inclusions and homomorphisms among modules

In this section, we assume

I, J are translation invariant, analytic P-ideals;

$\overrightarrow{a}$ is a base sequence adapted to both I and J.

Inclusion $\Rightarrow$ inclusion

Proposition

If $I \subseteq J$, then $I[\overrightarrow{a}] \subseteq J[\overrightarrow{a}]$.

Strong non-inclusion

I is **not included in J on intervals** if

$\exists$ lsc submeasures ϕ and ψ with $I = \mathrm{Exh}(\phi)$ and $J = \mathrm{Exh}(\psi)$, for which $\exists d > 0$ such that

$$\inf\{\phi(P) \mid P \text{ an interval in } \mathbb{N} \text{ with } \psi(P) \geq d\} = 0.$$

This property implies $I \not\subseteq J$.

Strong non-inclusion $\Rightarrow$ strong non-inclusion

I is **tall** if each infinite subset of $\mathbb{N}$ contains an infinite subset in I.

For a subset X of $\mathbb{R}$ and a real number c, we write

$$c\,X = \{cx \mid x \in X\}.$$

Theorem

I a tall ideal

If I is not included in J on intervals, then for each $c \neq 0$,

$$c\,I[\overrightarrow{a}] \not\subseteq J[\overrightarrow{a}].$$

About the proof:

$\forall \epsilon > 0$, $\exists w \in I[\overrightarrow{a}]$ such that

(a) $0 \leq w < \epsilon$;

(b) $\phi\big(j(w)\big) < \epsilon$;

(c) $\psi\big(j(c\,w)\big) > d$.

Assume for contradiction that $c\,I[\overrightarrow{a}] \subseteq J[\overrightarrow{a}]$, then

$$x \mapsto cx$$

defines a continuous homomorphism $I[\overrightarrow{a}] \to J[\overrightarrow{a}]$.

(a), (b) and (c) contradict continuity at 0.

Consequences on homomorphism

$\overrightarrow{a}$ is **uniform** if

$$\mathrm{pr}(\overrightarrow{a}) = \{p \mid \ p \text{ a prime and } p \mid a_n \text{ for some } n\}.$$

Lemma

I a tall ideal
$\overrightarrow{a}$ a uniform base sequence

If $f : I[\overrightarrow{a}] \to J[\overrightarrow{a}]$ is a continuous $\mathbb{Q}_{\overrightarrow{a}}$-module homomorphism, then there exists $c \in \mathbb{R}$ with $f(y) = c\,y$ for all $y \in I[\overrightarrow{a}]$.

Corollary

I a tall ideal

$\overrightarrow{a}$ a uniform base sequence

If I is not included in J on intervals, then each continuous $\mathbb{Q}_{\overrightarrow{a}}$-module homomorphism from $I[\overrightarrow{a}]$ to $J[\overrightarrow{a}]$ is identically equal to 0.

Proof of the main theorem

Family of P-ideals

$$P_k = \{n \in \mathbb{N} \mid 2^{k-1} \leq n < 2^k\}, k \in \mathbb{N}.$$

For $x \subseteq \mathbb{N}$, $A_x = \bigcup_{k \in x} P_k$,
and

$$\phi_x(a) = \sum_{n \in a \cap A_x} \frac{1}{2^n} + \sum_{n \in a \setminus A_x} \frac{1}{n}.$$

ϕ_x is a lsc submeasure.

Define

$$I_x = \mathrm{Exh}(\phi_x)$$

For each x, I_x is a translation invariant, analytic P-ideal that is tall.

$x \setminus y$ finite $\Longrightarrow I_x \subseteq I_y$

$x \setminus y$ infinite $\Longrightarrow I_x$ is not included in I_y on intervals

Proof of main theorem

R is given. Take $\overrightarrow{a}$ with three properties:

(i) $R = \mathbb{Q}_{\overrightarrow{a}}$;

(ii) $\overrightarrow{a}$ uniform;

(iii) $\overrightarrow{a}$ adapted to I_x for each $x \subseteq \mathbb{N}$.

For each x, $I_x[\overrightarrow{a}] \subseteq \mathbb{R}$ is a Polish R-module and the identity map is a continuous R-embedding.

If $x \setminus y$ is finite, then $I_x[\overrightarrow{a}] \subseteq I_y[\overrightarrow{a}]$. The inclusion map $I_x[\overrightarrow{a}] \to I_y[\overrightarrow{a}]$ is an R-module homomorphism and it is continuous by Pettis Theorem.

If $x \setminus y$ is infinite, then every continuous R-module homomorphsim from $I_x[\overrightarrow{a}] \to I_y[\overrightarrow{a}]$ is constantly 0.

Thank you!