Some Open Problems On Invariant Random Subgroups

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Invariant random subgroups

- Let G be a countable group and let

$$\mathrm{Sub}_G \subset \mathcal{P}(G) = \{\,0,1\,\}^G = 2^G$$

be the compact space of subgroups $H \leqslant G$.

- Note that $G \curvearrowright \mathrm{Sub}_G$ via conjugation: $H \overset{g}{\mapsto} g\,H\,g^{-1}$.

Definition (Abért)

A G-invariant Borel probability measure ν on Sub_G is called an invariant random subgroup or IRS.

Some Boring Examples

A Boring Example

If $N \trianglelefteq G$, then the Dirac measure δ_N is an IRS of G.

Another Boring Example

- Suppose that the IRS ν concentrates on a single conjugacy class $\mathcal{C} = \{\, gHg^{-1} \mid g \in G \,\}$ of subgroups of G.
- Then $\mathcal{C}$ is necessarily finite and hence $[\, G : N_G(H) \,] < \infty$.
- Furthermore, ν is the counting probability measure on $\mathcal{C}$.

Stabilizer distributions

Observation

- Suppose that $G \curvearrowright (Z, \mu)$ is a measure-preserving action on a Borel probability space.
- Let $f : Z \to \mathrm{Sub}_G$ be the G-equivariant map defined by $z \mapsto G_z = \{ g \in G \mid g \cdot z = z \}$.
- Then the stabilizer distribution $\nu = f_* \mu$ is an IRS of G, where if $B \subseteq \mathrm{Sub}_G$, then

$$\nu(B) = \mu(f^{-1}(B)) = \mu(\{ z \in Z \mid G_z \in B \}).$$

Theorem (Abért-Glasner-Virág 2012)

If ν is an IRS of G, then ν is the stabilizer distribution of a measure-preserving action $G \curvearrowright (Z, \mu)$.

Ergodicity

Definition

A measure-preserving action $G \curvearrowright (Z, \mu)$ is ergodic if $\mu(A) = 0, 1$ for every G-invariant μ-measurable subset $A \subseteq Z$.

Observation

If $G \curvearrowright (Z, \mu)$ is ergodic, then the corresponding stabilizer distribution ν is an ergodic IRS of G.

Theorem (Creutz-Peterson 2013)

If ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.

A less boring example of an IRS

Example

The group $\mathrm{Fin}(\mathbb{N}) = \{ g \in \mathrm{Sym}(\mathbb{N}) \mid \mathrm{supp}(g) \text{ is finite} \}$ of finite permutations of $\mathbb{N}$ has an ergodic IRS μ which does not concentrate on a single conjugacy class of subgroups.

- Let μ be the usual uniform product probability measure on $2^{\mathbb{N}}$.
- Then $\mathrm{Fin}(\mathbb{N})$ acts ergodically on $(2^{\mathbb{N}}, \mu)$ via the shift action $(g \cdot \xi)(n) = \xi(g^{-1}(n))$.
- For each $\xi \in 2^{\mathbb{N}}$ and $i = 0, 1$, let $B_i^\xi = \{ n \in \mathbb{N} \mid \xi(n) = i \}$.
- Then the stabilizer map is given by $\xi \overset{f}{\mapsto} \mathrm{Fin}(B_0^\xi) \times \mathrm{Fin}(B_1^\xi)$.
- Clearly the stabilizer distribution $\nu = f_* \mu$ does not concentrate on a single conjugacy class of subgroups of $\mathrm{Fin}(\mathbb{N})$.

Zero-one laws

Remark

- If ν is an ergodic IRS of a countable group G, then we obtain a corresponding zero-one law on Sub_G for the class of group-theoretic properties Φ such that the set $\{ H \in \mathrm{Sub}_G \mid H \text{ has property } \Phi \}$ is ν-measurable.
- Assuming suitable large cardinals, these include the properties with projective definitions and thus ν concentrates on a collection of subgroups which are quite difficult to distinguish between.
- In fact, until very recently, all of the known examples of ergodic IRSs ν had the property that ν concentrates on the subgroups of G of a fixed isomorphism type.

Zero-one laws continued

Remark

- It is well known that if $K \leqslant G$ is a subgroup, then

$$\{ H \in \mathrm{Sub}_G \mid H \cong K \}$$

  is a Borel subset of Sub_G.
- Hence, if ν is an ergodic IRS of G, then for each subgroup $K \leqslant G$,

$$\nu(\{ H \in \mathrm{Sub}_G \mid H \cong K \}) \in \{ 0, 1 \}.$$

Definition

An ergodic IRS ν of a countable group G is said to be diffuse if $\nu(\{ H \in \mathrm{Sub}_G \mid H \cong K \}) = 0$ for every subgroup $K \leqslant G$.

Theorem (Thomas)

There exist countable groups with diffuse IRSs.

Strenthening conjugacy to isomorphism

Main Lemma

If G is any countable group, then there exists a countable group N and a semidirect product $P = N \rtimes G$ such that for all $K_1, K_2 \in Sub_G$,

$$N \rtimes K_1 \cong N \rtimes K_2 \quad \Longleftrightarrow \quad (\exists g \in G)\; gK_1g^{-1} = K_2.$$

Proof of Theorem

- Let G be a countable group with an ergodic IRS μ which does not concentrate on a single conjugacy class of subgroups of G.
- Let N and $P = N \rtimes G$ be the countable groups given by the Lemma.
- Let $j : \mathrm{Sub}_G \to \mathrm{Sub}_P$ be the G-equivariant map defined by $j(K) = N \rtimes K$ and let $\nu = j_* \mu$ be the corresponding G-invariant ergodic probability measure on Sub_P.
- Since N acts trivially by conjugation on $j(\mathrm{Sub}_G)$, it follows that ν is P-invariant.
- Thus ν is an ergodic IRS of P.
- Furthermore, since the isomorphism classes on $j(\mathrm{Sub}_G)$ correspond to the conjugacy classes on Sub_G, it follows that ν is a diffuse IRS of P.

A Lemma of Burnside (1897)

Notation

- If S is a group and $s \in S$, then i_s denotes the corresponding inner automorphism, defined by $i_s(x) = sxs^{-1}$.
- $\mathrm{Inn}(S) = \{ i_s \mid s \in S \}$.

Lemma (Burnside)

Let S be a simple nonabelian group and let G, H be groups such that

$$\mathrm{Inn}(S) \leqslant G, H \leqslant \mathrm{Aut}(S).$$

If $\pi : G \to H$ is an isomorphism, then there exists $\varphi \in \mathrm{Aut}(S)$ such that $\pi(g) = \varphi g \varphi^{-1}$ for all $g \in G$.

Proof of Main Lemma

- By Fried-Kollár (1981), there exists a countably infinite field F such that $\mathrm{Aut}(F) = G$.
- By Schreier and van der Waerden (1928),

$$\mathrm{Aut}(\mathrm{PSL}(2,F)) = \mathrm{PGL}(2,F) \rtimes \mathrm{Aut}(F) = \mathrm{PGL}(2,F) \rtimes G.$$

- Suppose that $K_1, K_2 \in \mathrm{Sub}_G$.
- Clearly if K_1 and K_2 are conjugate subgroups of G, then $\mathrm{PGL}(2,F) \rtimes K_1 \cong \mathrm{PGL}(2,F) \rtimes K_2$.
- Conversely, suppose that

$$\pi : \mathrm{PGL}(2,F) \rtimes K_1 \to \mathrm{PGL}(2,F) \rtimes K_2$$

is an isomorphism.

- By Burnside's Lemma, there exists $h \in \mathrm{PGL}(2,F) \rtimes G$ such that $h(\mathrm{PGL}(2,F) \rtimes K_1)h^{-1} = \mathrm{PGL}(2,F) \rtimes K_2$.
- After factoring by $\mathrm{PGL}(2,F)$, we see that K_1 and K_2 are conjugate subgroups of G.

A Natural Example?

Problem

Find natural examples of groups G with diffuse IRSs.

Theorem (Raimbault)

- *Let T_4 be a triangle in the hyperbolic plane $\mathbb{H}^2$ with all three angles equal to $\pi/4$ and let G be the group of isometries generated by reflections in the faces of T_4.*
- *Then G is a finitely presented group with a diffuse IRS.*

A Flaw in the Abért-Glasner-Virág Construction?

Remark

If ν is an ergodic IRS of the countable group G, then the construction of Abért-Glasner-Virág realizes ν as the stabilizer distribution of a measure-preserving action $G \curvearrowright (X, \mu)$ such that the set

$$\{ x \in X \mid G_x = H \}$$

is uncountable for ν-a.e. $H \in \mathrm{Sub}_G$.

Question

Is this inevitable?

A Flaw in the Abért-Glasner-Virág Construction?

Proposition (Thomas)

Suppose that ν is an ergodic IRS of a countable group G and that $[\,N_G(H) : H\,] = \infty$ for ν-a.e. $H \in Sub_G$. If ν is the stabilizer distribution of a measure-preserving action $G \curvearrowright (\,X, \mu\,)$ on a Borel probability space, then the set $\{\, x \in X \mid G_x = H \,\}$ is uncountable for ν-a.e. $H \in Sub_G$.

Proof of Proposition

- If not, it follows that the set $\{ x \in X \mid G_x = H \}$ is countable for ν-a.e. $H \in \mathrm{Sub}_G$.
- Consider the Borel equivalence relation E on X defined by

$$x \, E \, y \quad \Longleftrightarrow \quad G_x = G_y.$$

- Then for μ-a.e. $x \in X$, the corresponding E-class $[\, x \,]_E$ is countable.
- Hence, after restricting to a Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$, we can suppose that $[\, x \,]_E$ is countable for every $x \in X$.
- Thus E is a smooth countable Borel equivalence relation on X.

Proof of Proposition

- It follows that $E' = E \cap E_G^X$ is also smooth.
- Since $[\, N_G(G_x) : G_x \,] = \infty$ and $G_x = gG_xg^{-1} = G_{g \cdot x}$ whenever $g \in N_G(G_x)$, it follows that every E'-class is infinite.
- Thus $E' \subseteq E_G^X$ is a smooth aperiodic Borel equivalence relation.
- By Dougherty-Jackson-Kechris, there does not exist a G-invariant Borel probability measure on X, which is a contradiction.

Question

Is this inevitable in the case when $[\, N_G(H) : H \,] < \infty$ for ν-a.e. $H \in Sub_G$?

Refining the Question

- Suppose that ν is an ergodic IRS of a countable group G such that $[\,N_G(H) : H\,] < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$.
- Then there exists an integer $n \geq 1$ such that $[\,N_G(H) : H\,] = n$ for ν-a.e. $H \in \mathrm{Sub}_G$.
- If $n = 1$, then ν is the stabilizer distribution of the ergodic action $G \curvearrowright (\,\mathrm{Sub}_G, \nu\,)$ and the corresponding stabilizer map

$$H \mapsto N_G(H) = H$$

is ν-a.e. injective.

Refining the Question

- Next suppose that $n > 1$ and that ν is the stabilizer distribution of the measure-preserving action $G \curvearrowright (X, \mu)$.
- If $x \in X$ and $g \in N_G(G_x)$, then $G_{g \cdot x} = gG_xg^{-1} = G_x$.
- It follows that for μ-a.e. $x \in X$, the stabilizer map $f : X \to \mathrm{Sub}_G$ is *n*-to-one on the orbit $G \cdot x$.
- Consequently, the stabilizer map f is μ-a.e. n-to-one iff the map

$$G \cdot x \mapsto \{ gG_xg^{-1} \mid g \in G \}$$

  is μ-a.e. injective.
- In this case, by restricting to a suitable G-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$, we obtain a measure-preserving action $G \curvearrowright (X_0, \mu)$ with stabilizer distribution ν such that the corresponding stabilizer map is n-to-one.

The Realization Problem

Open Problem

- Suppose that ν is an ergodic IRS of a countable group G and that $[\,N_G(H) : H\,] = n < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$.
- Is ν the stabilizer distribution of an ergodic action $G \curvearrowright (\,X, \mu\,)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one?

Theorem (Thomas)

This is true if G is amenable.

Towards a Solution of the Realization Problem

- Suppose that ν is an ergodic IRS of a countable group G and that $1 < [\, N_G(H) : H\,] = n < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$.
- Let $Z = \{\, H \in \mathrm{Sub}_G \mid [\, N_G(H) : H\,] = n \,\}$.
- And let $X = \{\, aH \mid H \in Z, a \in N_G(H) \,\}$.
- Then we can define a Borel probability measure μ on X by

$$\mu(B) = \int_Z \frac{|B \cap \{\, aH \mid a \in N_G(H) \,\}|}{n} \, d\nu(H).$$

Towards a Solution of the Realization Problem

- Let $c : E_G^Z \to G$ be a Borel map such that

$$c(H_1, H_2) H_1 c(H_1, H_2)^{-1} = H_2$$

  for each pair of conjugate subgroups H_1, $H_2 \in Z$.
- Then for each $g \in G$, we can define a corresponding Borel bijection $\pi_g : X \to X$ by

$$\begin{aligned} \pi_g(aH) &= c(H, gHg^{-1}) aHg^{-1} \\ &= g b_H^{-1} a g^{-1} (gHg^{-1}), \end{aligned}$$

  where $b_H \in N_G(H)$ is such that $g = c(H, gHg^{-1}) b_H$.
- It is clear that each π_g is μ-preserving.
- However, in order to ensure that these maps define a G-action, it is necessary to impose an extra hypothesis on the map $c : E_G^Z \to G$.

The Weak Cocycle Property

Definition

An IRS ν of a countable group G is said to have the weak cocycle property if there exists a G-invariant Borel subset $Z \subseteq Sub_G$ with $\nu(Z) = 1$ and a Borel map $c : E_G^Z \to G$ such that whenever H_1, H_2, $H_3 \in Z$ are conjugate subgroups of G, then:

- $c(H_1, H_2) H_1 c(H_1, H_2)^{-1} = H_2$; *and*
- $c(H_1, H_3)^{-1} c(H_2, H_3) c(H_1, H_2) \in H_1$.

Remark

The usual cocycle property has the stronger requirement that

$$c(H_1, H_3)^{-1} c(H_2, H_3) c(H_1, H_2) = 1.$$

The Realization Problem

Theorem (Thomas)

If ν is an ergodic IRS of a countable group G with the property that $[\,N_G(H) : H\,] = n < \infty$ for ν-a.e. $H \in Sub_G$, then the following conditions are equivalent:

(i) *ν has the weak cocycle property.*

(ii) *ν is the stabilizer distribution of an ergodic action $G \curvearrowright (\,X, \mu\,)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one.*

Corollary

If ν is an ergodic IRS of a countable amenable group G such that $[\,N_G(H) : H\,] = n < \infty$ for ν-a.e. $H \in Sub_G$, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (\,X, \mu\,)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one.

The Realization Problem

Theorem (Thomas)

There exist a countable group G with an ergodic IRS ν which does not have the weak cocycle property.

Remark

Unfortunately, in the above example, $[\,N_G(H) : H\,] = \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$.

Conjecture

There exists an ergodic IRS ν of a countable group G such that:

- $[\,N_G(H) : H\,] = n < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$; and
- ν is not the stabilizer distribution of an ergodic action $G \curvearrowright (\,X, \mu\,)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one.

Characters of countable groups

Definition

If G is a countable group, then $\chi : G \to \mathbb{C}$ is a character if the following conditions are satisfied:

(i) $\chi(1_G) = 1$.

(ii) $\chi(h\,g\,h^{-1}) = \chi(g)$ *for all* $g, h \in G$.

(iii) χ *is positive definite; i.e.*

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$$

for all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ *and* $g_1, \cdots, g_n \in G$.

Example

If $G \curvearrowright (\,Z, \mu\,)$ is a measure-preserving action on a Borel probability space, then $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ is a character.

Indecomposable characters

Definition

A character χ is indecomposable if it is impossible to express

$$\chi = r\chi_1 + (1 - r)\chi_2,$$

where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters.

Remark

Indecomposable characters of countable groups give rise (via the Gelfand-Naimark-Siegel construction) to the factor representations of finite type.

The Indecomposability Problem

Open Problem

Find necessary and sufficient conditions for the associated character $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ of an ergodic action $G \curvearrowright (\,Z, \mu\,)$ to be indecomposable.

Remark

I will soon formulate a conjecture.

Faithful actions

Definition

If $G \curvearrowright (Z, \mu)$ is a measure-preserving action of a countable group on a standard Borel probability space, then the μ-a.e. kernel is

$$K_\mu = \{ g \in G \mid \mu(\mathrm{Fix}_Z(g)) = 1 \};$$

and the action is said to be μ-a.e. faithful if $K_\mu = 1$.

Remark

If $K_\mu \neq 1$, then there exists a Borel subset $Z_0 \subseteq Z$ with $\mu(Z_0) = 1$ such that K_μ acts trivially on Z_0; and the induced action $G/K_\mu \curvearrowright (Z_0, \mu)$ is μ-a.e. faithful.

I.c.c. groups

Definition

A group G is said to have the infinite conjugacy class property, or to be an i.c.c. group, if the conjugacy class g^G of every nonidentity element $1 \neq g \in G$ is infinite.

Theorem

If $G \curvearrowright (Z, \mu)$ is a μ-a.e. faithful ergodic action of a countable non-i.c.c. group on a standard Borel probability space, then the associated character $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is decomposable.

A sufficient condition for decomposability

Theorem (Thomas)

If $G \curvearrowright (Z, \mu)$ is an ergodic action of a countable group on a standard Borel probability space and there exists a G-invariant Borel equivalence relation $E \subseteq E_G^Z$ such that $1 < [z]_E < \infty$ for μ-a.e. $z \in Z$, then the associated character $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is decomposable.

Conjecture

If $G \curvearrowright (Z, \mu)$ is a μ-a.e. faithful ergodic action of a countable i.c.c. group on a standard Borel probability space, then the following statements are equivalent:

(i) The associated character $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is decomposable.

(ii) There exists a G-invariant Borel equivalence relation $E \subseteq E_G^Z$ such that $1 < |[z]_E| < \infty$ for μ-a.e. $z \in Z$.

Strongly Simple Groups

Definition

The trivial IRSs of G are δ_1 and δ_G.

Definition

A countably infinite group G is said to be strongly simple if the only ergodic IRSs of G are δ_1 and δ_G.

Remark

Equivalently, G is strongly simple if for every ergodic action $G \curvearrowright (Z, \mu)$ on a standard Borel probability space, either:

(i) the action is μ-a.e. fixed-point-free; or

(ii) there exists a G-invariant point $z_0 \in Z$ with $\mu(\{ z_0 \}) = 1$.

Character Rigid Groups

Definition

The trivial characters of G are the regular character χ_{reg} and the constant character χ_{con}, where:

- $\chi_{reg}(g) = 0$ *for all* $1 \neq g \in G$*; and*
- $\chi_{con}(g) = 1$ *for all* $g \in G$.

Definition

A countably infinite group G is said to be character rigid if the only indecomposable characters of G are χ_{reg} and χ_{con}.

Character Rigid Groups

Theorem (Thomas-Tucker-Drob)

If the countably infinite group G is character rigid, then G is strongly simple.

Lemma (Ioana-Kechris-Tsankov)

If $G \curvearrowright (Z, \mu)$ is an ergodic measure-preserving action and there exists $r > 0$ such that $\mu(\mathrm{Fix}(g)) \geq r$ for all $g \in G$, then there exists a G-invariant point $z_0 \in Z$ with $\mu(\{z_0\}) = 1$.

Remark

This generalizes the result that if a finite group G acts transitively on a set Z with $|Z| > 1$, then some element $g \in G$ is fixed-point-free.

Proof of Theorem

- Suppose that G is character rigid and that $\nu \neq \delta_1, \delta_G$ is a nontrivial ergodic IRS of G.
- Then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.
- Let $\chi(g) = \mu(\text{Fix}_Z(g))$ be the associated character.
- Since G is character rigid, there exists $0 \leq r \leq 1$ such that $\chi = r\chi_{\text{con}} + (1-r)\chi_{\text{reg}}$.
- Since $\nu \neq \delta_1$, it follows that $r > 0$.
- Thus $\mu(\text{Fix}_Z(g)) \geq r$ for all $g \in G$ and so there exists a G-invariant point $z_0 \in Z$ with $\mu(\{z_0\}) = 1$.
- But then

$$\nu(\{G\}) = \mu(\{z \in Z \mid G = G_z\}) = 1$$

and so $\nu = \delta_G$, which is a contradiction.

Strongly Simple Groups

Conjecture

There exist a strongly simple group G which is not character rigid.

Conjecture

If K is a countable real closed field, then $G = SO(3, K)$ is a strongly simple group which is not character rigid.