Strong ergodicity phenomena for Bernoulli shifts of bounded algebraic dimension

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Dynamics and orbit equivalence relations

Given a continuous action $G \ltimes X$ of a Polish group $G$ on Polish space $X$ we let $E^G_X$ be the associated orbit equivalence relation:

$$xe^G_x' \iff \exists g \in G \ (g \cdot x = x').$$

**Question.** Which topological/dynamical properties of $G$ can be recovered from its orbit equivalence relations?
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Ways to measure the complexity of an orbit equivalence relation $(X, E^G_X)$:

1. **Its position within the Borel reduction hierarchy.**

We say that $(X, E)$ is **Borel reducible** to $(Y, F)$ and we write $E \leq_B F$ if there is a Borel map $f : X \to Y$ with $xEx' \iff f(x)Ff(x').
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2. **Its strong ergodic properties.**
   
   We say that $(X, E)$ is **strongly ergodic** with respect to $(Y, F)$ if for every Borel $f : X \to Y$ with $xEx' \implies f(x)Ff(x')$ there is a comeager $C \subseteq X$ so that for all $x, x' \in C$ we have that $f(x)Ff(x')$. 

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Dynamics and orbit equivalence relations

Theorem (Solecki)

Let $G$ be a Polish group. Then the following are equivalent:

1. $G$ is compact;
2. For all $G \curvearrowright X$ we have that $E_X^G$ is smooth, i.e., $(X, E_X^G) \leq_B (\mathbb{R}, =)$.
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Let $G$ be a Polish group. Then the following are equivalent:

1. $G$ is CLI;
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**Question (Kechris)**

Let $G$ be a Polish group which is not locally-compact. Does there exist some action $G \curvearrowright X$ so that $(X, E^G_X)$ is not essentially countable?
Polish permutation groups
Let $\text{Sym}(\mathbb{N})$ be the Polish group of all bijections $g: \mathbb{N} \to \mathbb{N}$ endowed with the pointwise convergence topology.

A **Polish permutation group** $P$ is any closed subgroup of $\text{Sym}(\mathbb{N})$. Such $P$ comes together with an action $P \curvearrowright \mathbb{N}$ with $(g, n) \mapsto g(n)$.
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The **Bernoulli shift** of $P$ is the induced action on $\mathbb{R}^\mathbb{N}$:

$$g \cdot (x_n : n \in \mathbb{N}) = (x_{g^{-1}(n)} : n \in \mathbb{N}).$$
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**Notation.** We denote by $E(P)$ the orbit equivalence relation of $P \curlywedge \mathbb{R}^\mathbb{N}$. We denote by $E_{\text{inj}}(P)$ the restriction of $E(P)$ to the $P$-invariant subset $\text{Inj}(\mathbb{N}, \mathbb{R})$ of $\mathbb{R}^\mathbb{N}$, consisting of all injective sequences.
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**Theorem (Kechris, Malicki, P., Zielinski)**

*If $P$ is not locally compact then $E_{\text{inj}}(P)$ is not essentially countable.*

*Similarly for when $P$ is non-compact or non-CLI.*
Algebraic dimension
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Let $P$ be a Polish permutation group.
For every $F \subseteq \mathbb{N}$ we have the **pointwise stabilizer**:

$$P_F := \{ g \in P : g(a) = a \text{ for all } a \in F \}$$
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$$[A]_P := \{b \in \mathbb{N} : \text{ the orbit } P_F \cdot b \text{ is finite, for some finite } F \subseteq A\}$$
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The assignment $\mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ with $A \mapsto [A]_P$ is a **closure operator**:

1. $A \subseteq [A]_P$;
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Definition

The algebraic dimension $\dim(P)$ of $P$ is the smallest $n \in \mathbb{N}$ so that for all $A \subseteq \mathbb{N}$ with $|A| = n + 1$, there is $a \in A$ so that $a \in [A \setminus \{a\}]_P$, if such $n$ exists. Otherwise, we write $\dim(P) = \infty$. 
Permutation groups of finite algebraic dimension

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Examples.

(1) Let $T_4$ be the infinite 4-regular tree:

![Diagram of a 4-regular tree]

Then $\dim(\text{Aut}(T_4)) = 1$
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Examples.

(2) Let $n \times T_4$ be the forest consisting of $n$-many infinite 4-regular trees:

Then $\dim(\text{Aut}(n \times T_4)) = n$
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(3) Let \( \mathbb{Q}^n \) be the \( n \)-dimensional \( \mathbb{Q} \)-vector space, then

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Remark. In examples (1),(2),(3) the permutation group $P$ happens to be locally-compact.
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Remark. In examples (1),(2),(3) the permutation group $P$ happens to be locally-compact.

This is a consequence of the fact that in all these examples the closure operator $A \mapsto [A]_P$ additionally satisfied the **exchange property**:

$$b \in [A \cup \{a\}]_P \setminus [A]_P \implies a \in [A \cup \{b\}]_P,$$

forming this way a *pre-geometry*. 
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There exist **non-locally compact** $P$ with $\dim(P) < \infty$. 

Bernoulli shifts and algebraic dimension

Let \( Q \) be a Polish permutation group. Recall the orbit equivalence relation:

\[ E_{\text{inj}}(Q), \text{ induced on the injective part of the Bernoulli shift } Q \curlyeqeq \text{Inj}(\mathbb{N}, \mathbb{R}). \]

**Question.** How much does \( E_{\text{inj}}(Q) \) remember of \( \dim(Q) \)?
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**Theorem (P., Shani)**

Let $P$ and $Q$ be Polish permutation groups and let $n \in \mathbb{N}$. Assume that:

1. $\dim(Q) \leq n$;
2. $P$ is locally-finite and $(n + 1)$–free.

Then, $E_{\text{inj}}(P)$ is strongly ergodic against $E_{\text{inj}}(Q)$. So, $E_{\text{inj}}(P) \not\preceq_B E_{\text{inj}}(Q)$. 
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- $P$ is **locally-finite** if for all finite $A \subseteq \mathbb{N}$ we have that $[A]_P$ is finite.
- $P$ is $(n + 1)$–free if for all finite $A \subseteq \mathbb{N}$ there are $g_0, g_1, \ldots, g_n \in P$ so that for all $i \leq n$ we have that $[g_iA]_P$ and $[\bigcup_{j: j \neq i} g_jA]_P$ are disjoint.
Some examples from Baldwin-Koerwien-Laskowski

$L_2 = \{f_0, f_1, f_2, \ldots\}$ consists of a sequence of $2$-ary function symbols. Let $M_2$ be the Fraïssé limit of the class $K_2$ of all finite $L$-structures $A$ s.t.

1. for all $a_0, a_1$ in $A$ and cofinitely many $n \in \mathbb{N}$, $f_n(a_0, a_1) = a_0$.
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Then $P_2 := \text{Aut}(M_2)$ is $2$-dimensional, locally-finite, $2$-free.
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Then $P_2 := \text{Aut}(\mathbb{M}_2)$ is 2-dimensional, locally-finite, 2-free.

Similarly, for every $n \geq 2$ we have $\mathcal{L}_n$, consisting of $n$-ary functions, and the corresponding Fraïssé class $\mathcal{K}_n$ whose Fraïssé limit satisfies:

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**Theorem.** (Kruckman, P.) If \( m \neq n \), then \( E_{\text{inj}}(P_m) \) and \( E_{\text{inj}}(P_n) \) are incomparable under \( * \)-reductions.
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**Theorem.** (Kruckman, P.) If \( m \neq n \), then \( E_{\text{inj}}(P_m) \) and \( E_{\text{inj}}(P_n) \) are incomparable under \(*\)-reductions.

**Corollary of our Main Theorem.** (P., Shani) If \( m \preceq n \), then \( E_{\text{inj}}(P_n) \) is strongly ergodic w.r.t. \( E_{\text{inj}}(P_m) \). In particular, we have that:

\[ E_{\text{inj}}(P_2) \preceq_B E_{\text{inj}}(P_3) \preceq_B E_{\text{inj}}(P_4) \preceq_B \cdots \]
Relationship with pinned cardinality

Let \((X, E)\) be an equivalence relation, \(\mathbb{P}\) be a poset, and \(\tau\) be a \(\mathbb{P}\)-name.

- \((\mathbb{P}, \tau)\) is an \(E\)-pin, if \(\mathbb{P} \times \mathbb{P}\) forces that \(\tau_l E \tau_r\).
- An \(E\)-pin \((\mathbb{P}, \tau)\) is trivial if there is \(x \in X\) so that \(\mathbb{P} \Vdash \check{x} E \tau\).
- \(E\) is pinned if all \(E\)-pins are trivial.

**Example.** Let \(E_{\text{inj}}(\text{Sym}(\mathbb{N}))\) be the injective part of \(\text{Sym}(\mathbb{N}) \sim \mathbb{R}^\mathbb{N}\): 
\[
(x_n : n \in \mathbb{N}) \, E_{\text{inj}}(\text{Sym}(\mathbb{N}))(y_n : n \in \mathbb{N}) \iff \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}
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Then \(E_{\text{inj}}(\text{Sym}(\mathbb{N}))\) is unpinned. Take \(\mathbb{P} := \text{Coll}(\mathbb{N}, \mathbb{R})\).
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Then \(E_{\text{inj}}(\text{Sym}(\mathbb{N}))\) is **unpinned**. Take \(\mathbb{P} := \text{Coll}(\mathbb{N}, \mathbb{R})\).

**Question.** (Kechris) Is \(E_{\text{inj}}(\text{Sym}(\mathbb{N}))\) the \(\leq_B\)-least unpinned E.R.?
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Zapletal exhibited unpinned: \(F_1 \preceq_B F_2 \preceq_B \cdots \preceq_B E_{\text{inj}}(\text{Sym}(\mathbb{N}))\)

The proof uses the theory of pinned cardinality.
Relationship with pinned cardinality

Let \((X, E)\) be an equivalence relation, \(\mathbb{P}\) be a poset, and \(\tau\) be a \(\mathbb{P}\)-name.

- \((\mathbb{P}, \tau)\) is an \(E\)-pin, if \(\mathbb{P} \times \mathbb{P}\) forces that \(\tau \leq E \tau\).
- An \(E\)-pin \((\mathbb{P}, \tau)\) is trivial if there is \(x \in X\) so that \(\mathbb{P} \vdash \exists x \, E \tau\).
- \(E\) is pinned if all \(E\)-pins are trivial.

**Example.** Let \(E_{\text{inj}}(\text{Sym}(\mathbb{N}))\) be the injective part of \(\text{Sym}(\mathbb{N}) \curvearrowright \mathbb{R}^\mathbb{N}\):
\[
(x_n : n \in \mathbb{N}) E_{\text{inj}}(\text{Sym}(\mathbb{N}))(y_n : n \in \mathbb{N}) \iff \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}
\]
Then \(E_{\text{inj}}(\text{Sym}(\mathbb{N}))\) is unpinned. Take \(\mathbb{P} := \text{Coll}(\mathbb{N}, \mathbb{R})\).

**Question.** (Kechris) Is \(E_{\text{inj}}(\text{Sym}(\mathbb{N}))\) the \(\leq_B\)-least unpinned E.R. ?

Zapletal exhibited unpinned: \(F_1 \not\leq_B F_2 \not\leq_B \cdots \not\leq_B E_{\text{inj}}(\text{Sym}(\mathbb{N}))\)
The proof uses the theory of pinned cardinality.
The minimum of the above sequence and the minimum of our sequence:
\(E_{\text{inj}}(P_2) \not\leq_B E_{\text{inj}}(P_3) \not\leq_B E_{\text{inj}}(P_4) \not\leq_B \cdots\) have pinned cardinality \(\aleph_1\).
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**Corollary.** (P., Shani) \(E_{\text{inj}}(P_2)) \not\leq_B F_1\).
Relationship with pinned cardinality

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Zapletal exhibited unpinned: 
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F_1 \preceq_B F_2 \preceq_B \cdots \preceq_B E_{\text{inj}}(\text{Sym}(\mathbb{N}))
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The minimum of the above sequence and the minimum of our sequence:
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**Corollary.** (P., Shani) \(E_{\text{inj}}(P_2)) \preceq_B F_1\).

**Question.** What about the converse? Is there a nice basis for the class of unpinned equivalence relations under Borel reductions?
1 Some words on the proof
Main theorem

Theorem (P., Shani)

Let $P$ and $Q$ be Polish permutation groups and let $n \in \mathbb{N}$. Assume that:

1. $\dim(Q) \leq n$;
2. $P$ is locally-finite and $(n + 1)$–free.

Then, $E_{\text{inj}}(P)$ is strongly ergodic against $E_{\text{inj}}(Q)$. So, $E_{\text{inj}}(P) \not\preceq_B E_{\text{inj}}(Q)$.
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The proof employs/builds on symmetric model techniques.
Theorem (Shani)

Suppose $E$ and $F$ are Borel equivalence relations on $X$ and $Y$ respectively and $x \mapsto N^x$ and $y \mapsto M^y$ be classifications by countable structures of $E$ and $F$ respectively. Then, the following are equivalent.

1. For every Borel homomorphism $f : (X_0, E) \to (Y, F)$, where $X_0 \subseteq X$ is non-meager, $f$ maps a non-meager set into a single $F$-class;

2. If $x \in X$ is Cohen-generic over $V$ and $M$ is a potential $F$-invariant in $V(N^x)$, definable from $N^x$ and parameters in $V$, then $M \in V$. 
The basic Cohen model

Recall Cohen’s proof that $\text{ZF} + \neg\text{AC}$ is consistent.
The basic Cohen model

Recall Cohen’s proof that $ZF + \neg AC$ is consistent.

Let $\mathbb{P}$ be the forcing which adds a countable sequence of Cohen reals:

$$(x_n^G : n \in \mathbb{N})$$
The basic Cohen model

Recall Cohen’s proof that $\text{ZF} + \neg\text{AC}$ is consistent. Let $\mathbb{P}$ be the forcing which adds a countable sequence of Cohen reals:

$$ (x^G_n : n \in \mathbb{N}) $$

Between $V$ and $V[G]$ there is the intermediate “symmetric model”:

$$ V(\{x^G_n : n \in \mathbb{N}\}) $$

This can be defined in a number of equivalent ways:

- it consists of the realization of all symmetric names ($\text{Sym}(\mathbb{N}) \leq \mathbb{P}$);
- it is the smallest ZF-extension of $V$ in $V[G]$ containing $\{x^G_n : n \in \mathbb{N}\}$. 
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**Theorem.** (Cohen) In $V(\{x^G_n\})$ there is no injection $\mathbb{N} \to \{x^G_n : n \in \mathbb{N}\}$
The basic Cohen model

Recall Cohen’s proof that $\text{ZF} + \neg \text{AC}$ is consistent. Let $\mathbb{P}$ be the forcing which adds a countable sequence of Cohen reals:

$$\left( x_n^G : n \in \mathbb{N} \right)$$

Between $V$ and $V[G]$ there is the intermediate “symmetric model”:

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**Theorem.** (Cohen) In $V(\{x_n^G\})$ there is no injection $\mathbb{N} \to \{x_n^G : n \in \mathbb{N}\}$

**Lemma.** (Existence of supports) For any $S \in V(\{x_n^G\})$ with $S \subseteq V$ there is a finite $F \subseteq \{x_n^G : n \in \mathbb{N}\}$ so that $S \in V[F]$.
Symmetric models from permutation groups

In the basic Cohen model the action $\text{Sym}(\mathbb{N}) \curvearrowright \mathbb{P}$ gave:

$$(x_n^G : n \in \mathbb{N}) \mapsto \{x_n^G : n \in \mathbb{N}\}$$
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If $P$ is a Polish permutation group, then the Bernoulli shift action $P \curvearrowright \mathbb{R}^\mathbb{N}$ is essentially $P \curvearrowright \mathbb{P}$, and the generic $(x_n^G : n \in \mathbb{N})$ is injective.
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$$(x_n^G : n \in \mathbb{N}) \mapsto \mathcal{N}^G$$

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We have the intermediate symmetric model $V \subseteq V(\mathcal{N}^G) \subseteq V[G]$:

- it consists of the realization of all symmetric names $(P \curvearrowright \mathbb{P})$;
- it is the smallest ZF-extension of $V$ in $V[G]$ containing $\mathcal{N}^G$. 
Symmetric models from permutation groups

In the basic Cohen model the action \( \text{Sym}(\mathbb{N}) \ltimes \mathbb{P} \) gave:

\[
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\]

If \( P \) is a Polish permutation group, then the Bernoulli shift action \( P \ltimes \mathbb{R}^\mathbb{N} \) is essentially \( P \ltimes \mathbb{P} \), and the generic \( (x_n^G : n \in \mathbb{N}) \) is injective. But \( P = \text{Aut}(\mathcal{N}) \) for some countable structure \( \mathcal{N} \) on \( \mathbb{N} \). We have

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(x_n^G : n \in \mathbb{N}) \mapsto \mathcal{N}^G
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where \( \mathcal{N}^G \) is the structure \( \mathcal{N} \) copied on \( \{x_n^G : n \in \mathbb{N}\} \).

We have the intermediate symmetric model \( V \subseteq V(\mathcal{N}^G) \subseteq V[G] \):

- it consists of the realization of all \textbf{symmetric} names \( (P \ltimes \mathbb{P}) \);
- it is the smallest ZF-extension of \( V \) in \( V[G] \) containing \( \mathcal{N}^G \).

**Lemma ((P., Shani) Existence of supports)**

\textit{If \( P \) is a \textbf{locally-finite} Polish permutation group, then for all \( S \in V(\mathcal{N}^G) \) with \( S \subseteq V \) there is a finite \( F \subseteq \{x_n^G : n \in \mathbb{N}\} \) so that \( S \in V[F] \).}
To conclude:

**Theorem (Shani)**

Suppose $E$ and $F$ are Borel equivalence relations on $X$ and $Y$ respectively and $x \mapsto \mathcal{N}^x$ and $y \mapsto \mathcal{M}^y$ be classifications by countable structures of $E$ and $F$ respectively. Then, the following are equivalent.

1. For every Borel homomorphism $f : (X_0, E) \to (Y, F)$, where $X_0 \subseteq X$ is non-meager, $f$ maps a non-meager set into a single $F$-class;
2. If $x \in X$ is Cohen-generic over $V$ and $\mathcal{M}$ is a potential $F$-invariant in $V(N^x)$, definable from $N^x$ and parameters in $V$, then $\mathcal{M} \in V$.

In the case of the Bernoulli shifts, we have that $P = \text{Aut}(\mathcal{N})$ and $Q = \text{Aut}(\mathcal{M})$ for countable structures $\mathcal{M}$ and $\mathcal{N}$. So we have that:

$$P \preccurlyeq \text{Inj}(\mathbb{N}, \mathbb{R}) \text{ is classified by } (x_n : n \in \mathbb{N}) \mapsto \text{“} \mathcal{N} \text{ on } \{x_n : n \in \mathbb{N}\} \text{”}$$

$$Q \preccurlyeq \text{Inj}(\mathbb{N}, \mathbb{R}) \text{ is classified by } (y_n : n \in \mathbb{N}) \mapsto \text{“} \mathcal{M} \text{ on } \{y_n : n \in \mathbb{N}\} \text{”}$$
Thank you!