Strong ergodicity phenomena for Bernoulli shifts of bounded algebraic dimension

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Dynamics and orbit equivalence relations

Given a continuous action $G \curvearrowright X$ of a Polish group G on Polish space X we let E^G_X be the associated **orbit equivalence relation**:

$$xE^G_X x' \iff \exists g \in G \, (g \cdot x = x').$$

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Ways to measure the complexity of an orbit equivalence relation (X, E_X^G):

(1) Its position within the Borel reduction hierarchy.

We say that (X, E) is **Borel reducible** to (Y, F) and we write $E \leq_B F$ if there is a Borel map $f \colon X \to Y$ with $xEx' \iff f(x)Ff(x')$.

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(2) Its strong ergodic properties.

We say that (X, E) is **strongly ergodic** with respect to (Y, F) if for every Borel $f \colon X \to Y$ with $xEx' \implies f(x)Ff(x')$ there is a comeager $C \subseteq X$ so that for all $x, x' \in C$ we have that $f(x)Ff(x')$.

Dynamics and orbit equivalence relations

Theorem (Solecki)

Let G be a Polish group. Then the following are equivalent:

1. *G is* **compact***;*
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Question (Kechris)

Let G be a Polish group which is **not locally-compact***. Does there exist some action $G \curvearrowright X$ so that (X, E_X^G) is not* **essentially countable***?*

Polish permutation groups

Let $\mathrm{Sym}(\mathbb{N})$ be the Polish group of all bijections $g\colon \mathbb{N} \to \mathbb{N}$ endowed with the pointwise convergence topology.

A **Polish permutation group** P is any closed subgroup of $\mathrm{Sym}(\mathbb{N})$. Such P comes together with an action $P \curvearrowright \mathbb{N}$ with $(g, n) \mapsto g(n)$.

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The **Bernoulli shift** of P is the induced action on $\mathbb{R}^{\mathbb{N}}$:

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Notation. We denote by $E(P)$ the orbit equivalence relation of $P \curvearrowright \mathbb{R}^{\mathbb{N}}$.
We denote by $E_{\mathrm{inj}}(P)$ the restriction of $E(P)$ to the P-invariant subset $\mathrm{Inj}(\mathbb{N}, \mathbb{R})$ of $\mathbb{R}^{\mathbb{N}}$, consisting of all injective sequences.

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Theorem (Kechris, Malicki, P., Zielinski)

If P is **not locally compact** *then $E_{\mathrm{inj}}(P)$ is not essentially countable. Similarly for when P is non-compact or non-CLI.*

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Definition

The **algebraic dimension** $\dim(P)$ of P is the smallest $n \in \mathbb{N}$ so that for all $A \subseteq \mathbb{N}$ with $|A| = n + 1$, there is $a \in A$ so that $a \in [A \setminus \{a\}]_P$, if such n exists. Otherwise, we write $\dim(P) = \infty$.

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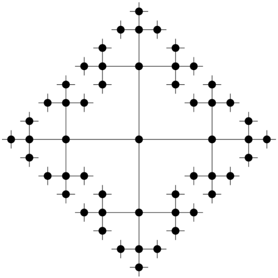
Examples.

(1) Let T_4 be the infinite 4-regular tree:



Then $\dim(\mathrm{Aut}(T_4)) = 1$

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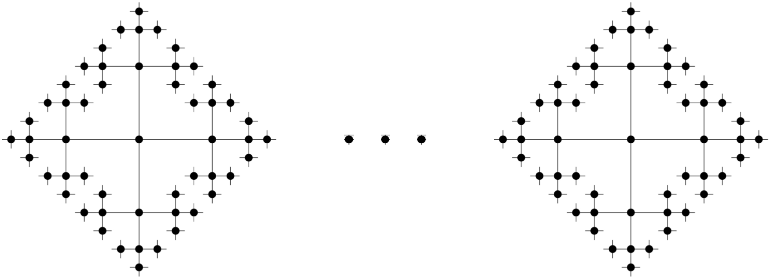
(2) Let $n \times T_4$ be the forest consisting of n-many infinite 4-regular trees:



Then $\dim(\mathrm{Aut}(n \times T_4)) = n$

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(3) Let $\mathbb{Q}^n$ be the n-dimensional $\mathbb{Q}$-vector space, then

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This is a consequence of the fact that in all these examples the closure operator $A \mapsto [A]_P$ additionally satisfied the **exchange property**:

$$b \in [A \cup \{a\}]_P \setminus [A]_P \implies a \in [A \cup \{b\}]_P,$$

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There exist **non-locally compact** P with $\dim(P) < \infty$.

Bernoulli shifts and algebraic dimension

Let Q be a Polish permutation group. Recall the orbit equivalence relation:

$E_{\mathrm{inj}}(Q)$, induced on the injective part of the Bernoulli shift $Q \curvearrowright \mathrm{Inj}(\mathbb{N}, \mathbb{R})$.

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Theorem (P., Shani)

Let P and Q be Polish permutation groups and let $n \in \mathbb{N}$. Assume that:

1. $\dim(Q) \leq n$;
2. P *is* **locally-finite** *and* $(n+1)$**–free**.

Then, $E_{\mathrm{inj}}(P)$ *is strongly ergodic against* $E_{\mathrm{inj}}(Q)$. *So,* $E_{\mathrm{inj}}(P) \not\leq_B E_{\mathrm{inj}}(Q)$.

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- P is **locally-finite** if for all finite $A \subseteq \mathbb{N}$ we have that $[A]_P$ is finite.
- P is $(n+1)$**–free** if for all finite $A \subseteq \mathbb{N}$ there are $g_0, g_1, \ldots, g_n \in P$ so that for all $i \leq n$ we have that $[g_i A]_P$ and $[\bigcup_{j:j\neq i} g_j A]_P$ are disjoint.

Some examples from Baldwin-Koerwien-Laskowski

$\mathcal{L}_{\mathbf{2}} = \{f_0, f_1, f_2, \ldots\}$ consists of a sequence of $\mathbf{2}$-ary function symbols.
Let $\mathbb{M}_{\mathbf{2}}$ be the Fraïssé limit of the class $\mathcal{K}_{\mathbf{2}}$ of all finite $\mathcal{L}$-structures $\mathbb{A}$ s.t.

1. for all a_0, a_1 in $\mathbb{A}$ and cofinitely many $n \in \mathbb{N}$, $f_n(a_0, a_1) = a_0$.
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Then $P_{\mathbf{2}} := \mathrm{Aut}(\mathbb{M}_{\mathbf{2}})$ is $\mathbf{2}$-dimensional, locally-finite, $\mathbf{2}$-free.

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Similarly, for every $\boldsymbol{n} \geq \mathbf{2}$ we have $\mathcal{L}_{\boldsymbol{n}}$, consisting of $\boldsymbol{n}$-ary functions, and the corresponding Fraïssé class $\mathcal{K}_{\boldsymbol{n}}$ whose Fraïssé limit satisfies:

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Corollary of our Main Theorem. (P., Shani) If $m \lneq n$, then $E_{\mathrm{inj}}(P_{\boldsymbol{n}}))$ is strongly ergodic w.r.t. $E_{\mathrm{inj}}(P_{\boldsymbol{m}}))$. In particular, we have that:

$$E_{\mathrm{inj}}(P_{\mathbf{2}})) \lneq_B E_{\mathrm{inj}}(P_{\mathbf{3}})) \lneq_B E_{\mathrm{inj}}(P_{\mathbf{4}})) \lneq_B \cdots$$

Relationship with pinned cardinality

Let (X,E) be an equivalence relation, $\mathbb{P}$ be a poset, and τ be a $\mathbb{P}$-name.

- $(\mathbb{P},\tau)$ is an E-pin, if $\mathbb{P}\times\mathbb{P}$ forces that $\tau_l E \tau_r$.
- An E-pin $(\mathbb{P},\tau)$ is trivial if there is $x\in X$ so that $\mathbb{P}\Vdash \check{x} E \tau$
- E is **pinned** if all E-pins are trivial.

Example. Let $E_{\text{inj}}(\mathrm{Sym}(\mathbb{N}))$ be the injective part of $\mathrm{Sym}(\mathbb{N}) \curvearrowright \mathbb{R}^{\mathbb{N}}$:
$(x_n\colon n\in\mathbb{N}) E_{\text{inj}}(\mathrm{Sym}(\mathbb{N}))(y_n\colon n\in\mathbb{N}) \iff \{x_n\colon n\in\mathbb{N}\}=\{y_n\colon n\in\mathbb{N}\}$
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Question. (Kechris) Is $E_{\text{inj}}(\mathrm{Sym}(\mathbb{N}))$ the $\leq_B$-least unpinned E.R. ?

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Zapletal exhibited unpinned: $F_1 \lneq_B F_2 \lneq_B \overbrace{\cdots}^{\aleph_1} \lneq_B E_{\mathrm{inj}}(\mathrm{Sym}(\mathbb{N}))$
The proof uses the theory of pinned cardinality.

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Let (X, E) be an equivalence relation, $\mathbb{P}$ be a poset, and τ be a $\mathbb{P}$-name.

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Example. Let $E_{\mathrm{inj}}(\mathrm{Sym}(\mathbb{N}))$ be the injective part of $\mathrm{Sym}(\mathbb{N}) \curvearrowright \mathbb{R}^{\mathbb{N}}$:
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Question. (Kechris) Is $E_{\mathrm{inj}}(\mathrm{Sym}(\mathbb{N}))$ the $\leq_B$-least unpinned E.R. ?

Zapletal exhibited unpinned: $F_1 \lneq_B F_2 \lneq_B \overbrace{\cdots}^{\aleph_1} \lneq_B E_{\mathrm{inj}}(\mathrm{Sym}(\mathbb{N}))$
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The minimum of the above sequence and the minimum of our sequence:
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Corollary. (P., Shani) $E_{\mathrm{inj}}(P_{\mathbf{2}})) \lneq_B F_1$.

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Question. What about the converse? Is there a nice basis for the class of unpinned equivalence relations under Borel reductions?

Table of Contents

Main theorem

Theorem (P., Shani)

Let P and Q be Polish permutation groups and let $n \in \mathbb{N}$. Assume that:

1. $\dim(Q) \leq n$;
2. P *is locally-finite and* $(n+1)$*–free.*

Then, $E_{\mathrm{inj}}(P)$ *is strongly ergodic against* $E_{\mathrm{inj}}(Q)$*. So,* $E_{\mathrm{inj}}(P) \not\leq_B E_{\mathrm{inj}}(Q)$*.*

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The proof employs/builds on **symmetric model techniques**.

The dictionary

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Suppose E and F are Borel equivalence relations on X and Y respectively and $x \mapsto \mathcal{N}^x$ and $y \mapsto \mathcal{M}^y$ be classifications by countable structures of E and F respectively. Then, the following are equivalent.

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Lemma. (*Existence of supports*) For any $S \in V(\{x_n^G\})$ with $S \subseteq V$ there is a finite $F \subseteq \{x_n^G : n \in \mathbb{N}\}$ so that $S \in V[F]$.

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To conclude:

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In the case of the Bernoulli shifts, we have that $P = \mathrm{Aut}(\mathcal{N})$ and $Q = \mathrm{Aut}(\mathcal{M})$ for countable structures $\mathcal{M}$ and $\mathcal{N}$. So we have that:

$P \curvearrowright \mathrm{Inj}(\mathbb{N}, \mathbb{R})$ is classified by $(x_n : n \in \mathbb{N}) \mapsto$ "$\mathcal{N}$ on $\{x_n : n \in \mathbb{N}\}$"

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Thαnk you!