The number of ergodic models of an infinitary sentence

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Span of the talk

1. Ergodic models of $\mathscr{L}_{\omega_1\omega}$-sentences
2. The ergodic spectrum for $\mathscr{L}_{\omega_1\omega}$-sentences
3. Highly homogeneous structures
4. The range of the spectrum function
5. Two questions

Ergodic models of $\mathscr{L}_{\omega_1\omega}$-sentences

The measurable space Str_L

Throughout this talk, L is a countable language.

Write **$\mathbf{Str}_L$** for the measurable space consisting of L-structures with underlying set $\mathbb{N}$, equipped with the Borel σ-algebra generated by subbasic open sets

$$\{\mathcal{M} \in \mathrm{Str}_L : \mathcal{M} \models R(\bar{a})\} \quad \text{and} \quad \{\mathcal{M} \in \mathrm{Str}_L : \mathcal{M} \models \neg R(\bar{a})\}$$

for relation symbols $R \in L$ and tuples $\bar{a} \in \mathbb{N}$ with $|\bar{a}| = \mathrm{arity}(R)$; and similarly for constant and function symbols in L.

For a sentence ϑ of $\mathscr{L}_{\omega_1\omega}(L)$, define the **extent** of ϑ in Str_L to be:

$$[\![\boldsymbol{\vartheta}]\!] := \{\mathcal{M} \in \mathrm{Str}_L : \mathcal{M} \models \vartheta\}.$$

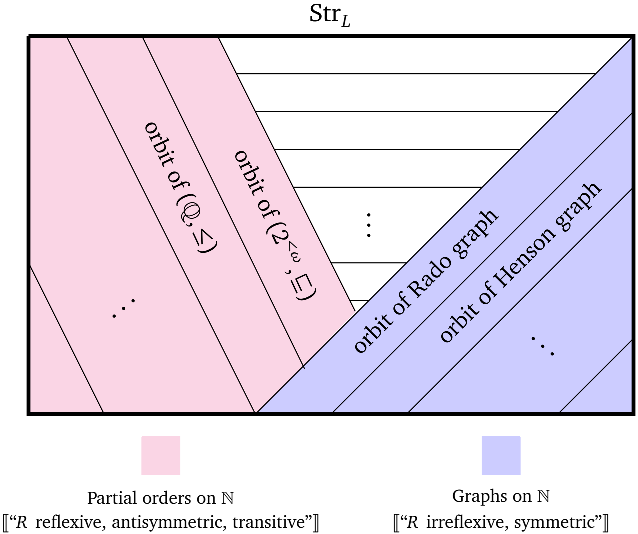
The logic action on Str_L

The group S_∞ of permutations of $\mathbb{N}$ acts on Str_L via the **logic action**, by permuting the underlying set: For $g \in S_\infty$ and $\mathscr{M} \in \mathrm{Str}_L$, the structure $g \cdot \mathscr{M} \in \mathrm{Str}_L$ is obtained by relabelling the elements of $\mathscr{M}$ according to g.

- ⋆ Orbits of the logic action are precisely the isomorphism classes of L-structures, i.e., extents of Scott sentences.
- ⋆ The extent of any $\mathscr{L}_{\omega_1\omega}(L)$-sentence is Borel; and it is also invariant under the logic action, i.e., for any $\mathscr{L}_{\omega_1\omega}(L)$-sentence ϑ and $g \in S_\infty$,

$$g \cdot [\![\vartheta]\!] = [\![\vartheta]\!].$$

Example. $L = \{R\}$, where R is a binary relation symbol.



Ergodic invariant probability measures on Str_L

A probability measure μ on Str_L is **(S_∞-) invariant** when the logic action does not change the μ-measure of a Borel subset of Str_L, i.e., when $\mu(X) = \mu(g \cdot X)$ for every Borel subset X of Str_L and every $g \in S_\infty$.

Further, such a μ is **ergodic** when, for any Borel subset X of Str_L such that $\mu\big(X \triangle g \cdot X\big) = 0$ for all $g \in S_\infty$, we have either $\mu(X) = 0$ or $\mu(X) = 1$.

- ⋆ Fact: The set of invariant probability measures on Str_L is a convex set. Its extreme points are the ergodic invariant probability measures; any invariant probability measure on Str_L is a mixture of ergodic ones.

  Thus, without loss of generality, we may consider only the ergodic invariant probability measures on Str_L.

Ergodic models of an $\mathscr{L}_{\omega_1\omega}$-sentence

μ an ergodic invariant probability measure on Str_L.

- ⋆ Since extents of sentences are invariant under the logic action, for any sentence ϑ of $\mathscr{L}_{\omega_1\omega}(L)$, we have that $\mu\left([\![\vartheta]\!]\right)$ equals either 0 or 1, i.e., ϑ almost surely holds or almost surely does not hold with respect to μ.

Define the **theory** of μ to be:

$$\mathbf{Th}(\boldsymbol{\mu}) := \left\{\vartheta \text{ an } \mathscr{L}_{\omega_1\omega}(L)\text{-sentence} : \mu\left([\![\vartheta]\!]\right) = 1\right\}.$$

- ⋆ $\mathrm{Th}(\mu)$ is complete and countably satisfiable (by ergodicity and σ-additivity, respectively).

Hence we call an ergodic invariant probability measure μ on Str_L an **ergodic structure**. We say μ is an **ergodic model** of ϑ when $\vartheta \in \mathrm{Th}(\mu)$.

The ergodic spectrum for $\mathscr{L}_{\omega_1\omega}$-sentences

The ergodic spectrum

We define the **ergodic spectrum I** to be the function on $\mathscr{L}_{\omega_1\omega}(L)$-sentences given by: $I(\vartheta)$ is the number of ergodic models of ϑ.

Main Question. What values can $I(\vartheta)$ take?

- Note that $I(\vartheta) \leq 2^{\aleph_0}$, as L is countable.
- Note also that if $\vartheta \models \xi$, then $I(\vartheta) \leq I(\xi)$.

Pop Quiz. What is the value of $I(\vartheta)$ when ϑ is:

- a Scott sentence for $(\mathbb{Z}, \leq)$
- a Scott sentence for $(\mathbb{Q}, \leq)$
- a Scott sentence for the Rado graph
- the model companion of the theory of $\aleph_0$-many irreflexive, symmetric binary relations

Trivial definable closure and $I(\vartheta)$

A criterion for existence of an ergodic model

Recall the model-theoretic notion of trivial definable closure for a structure. We extend this notion to $\mathscr{L}_{\omega_1\omega}$-sentences.

An $\mathscr{L}_{\omega_1\omega}(L)$-sentence ϑ has **trivial definable closure** when, for any countable fragment F of $\mathscr{L}_{\omega_1\omega}$ and complete F-theory Σ such that $\vartheta \in \Sigma \subseteq F$, there is no formula in F that uniformly witnesses non-trivial $\mathscr{L}_{\omega_1\omega}(L)$-definable closure in all models of Σ, i.e., there is no formula $\varphi(\bar{x}, y)$ in F, with $|\bar{x}| := n$, such that

$$\Sigma \models \exists \bar{x}\, \exists^{=1} y \left(\left(\wedge_{i=1}^{n} y \neq x_i \right) \wedge \varphi(\bar{x}, y) \right).$$

Theorem (Ackerman–Freer–P. 2017). For any $\mathscr{L}_{\omega_1\omega}(L)$-sentence ϑ, $I(\vartheta) > 0$ if and only if ϑ has trivial definable closure.

Proper ergodicity and $I(\vartheta)$

A sufficient condition for attaining the maximum

An ergodic structure μ is **properly ergodic** when μ is not an ergodic model of any Scott sentence, i.e., when $\vartheta \notin \mathrm{Th}(\mu)$ holds for any Scott sentence ϑ.

- ⋆ A properly ergodic structure on Str_L assigns measure 0 to every isomorphism class of structures in Str_L.
- ⋆ If μ is an ergodic structure that is not properly ergodic, then $\mathrm{Th}(\mu)$ contains exactly one Scott sentence, to which it is equivalent.

Theorem (Ackerman–Freer–Kruckman–P. 2017). If an $\mathscr{L}_{\omega_1\omega}(L)$-sentence ϑ has a properly ergodic model, then $I(\vartheta) = 2^{\aleph_0}$.

Pop Quiz, Redux. What is the value of $I(\vartheta)$ when ϑ is:

- a Scott sentence for $(\mathbb{Z}, \leq)$
- a Scott sentence for $(\mathbb{Q}, \leq)$
- a Scott sentence for the Rado graph
- the model companion of the theory of $\aleph_0$-many irreflexive, symmetric binary relations

Pop Quiz, Redux. What is the value of $I(\vartheta)$ when ϑ is:

- a Scott sentence for $(\mathbb{Z}, \leq)$
- a Scott sentence for $(\mathbb{Q}, \leq)$
- a Scott sentence for the Rado graph
- the model companion of the theory of $\aleph_0$-many irreflexive, symmetric binary relations

Extra Credit. What is the value of $I(\vartheta)$ when ϑ is:

- a disjunction of Scott sentences for $(\mathbb{Z}, \leq)$ and for the pure set $\mathbb{N}$
- a disjunction of Scott sentences for $(\mathbb{Q}, \leq)$ and for the pure set $\mathbb{N}$
- a disjunction of Scott sentences for the Rado graph and for the pure set $\mathbb{N}$

Highly homogeneous structures

High homogeneity

The key property in the classification of $I(\vartheta)$

A structure $\mathscr{M}$ is **highly homogeneous** when, for any finite subsets A, B of $\mathscr{M}$ with $|A| = |B|$, there is an automorphism σ of $\mathscr{M}$ such that $\sigma[A] = B$.

- ⋆ Any highly homogeneous structure is $\aleph_0$-categorical, by the Engeler–Ryll-Nardzewski–Svenonius Theorem.

Key Observation. The following $\mathscr{L}_{\omega_1\omega}(L)$-sentence defines high homogeneity among countable L-structures:

$$\mathfrak{H}\mathfrak{H} := \bigwedge_{n<\omega} \Big(\forall x_0, \ldots, x_{n-1}, y_o, \ldots, y_{n-1} \big(x_i \text{ distinct, } y_i \text{ distinct } \rightarrow \bigvee_{\sigma \in S_n} \bigwedge_{\psi \in \mathscr{L}_{\omega\omega}(L)} \psi(x_0, \ldots, x_{n-1}) \leftrightarrow \psi(y_{\sigma(0)}, \ldots, y_{\sigma(n-1)}) \big) \Big)$$

Peter Cameron's classification

The highly homogeneous structures are essentially the reducts of $(\mathbb{Q}, <)$

Theorem. (Cameron, 1976) Up to isomorphism, the countably infinite highly homogeneous structures are the reducts of $(\mathbb{Q}, \leq)$, namely, the structures interdefinable with one of the following:

- $\mathbb{Q}$ as a pure set
- $(\mathbb{Q}, \leq)$, the rational linear order
- $(\mathbb{Q}, B)$, where B is the ternary **betweenness** relation
- $(\mathbb{Q}, K)$, where B is the ternary **circular order** relation
- $(\mathbb{Q}, S)$, where S is the quaternary **separation** relation

High homogeneity and $I(\vartheta)$ for ϑ a Scott sentence

High homogeneity characterises unique ergodicity for Scott sentences.

Theorem (Ackerman–Freer–Kwiatkowska–P. 2016). Let $\mathscr{M} \in \mathrm{Str}_L$ and ϑ a Scott sentence for $\mathscr{M}$. Exactly one of the following holds.

- ϑ has non-trivial definable closure, in which case $I(\vartheta) = 0$
- $\mathscr{M}$ is highly homogeneous, in which case $I(\vartheta) = 1$
- $I(\vartheta) = 2^{\aleph_0}$

High homogeneity and $I(\vartheta)$ for arbitrary ϑ

Recall the $\mathscr{L}_{\omega_1\omega}(L)$-sentence $\mathfrak{H}\mathfrak{H}$: A structure $\mathscr{M} \in \mathrm{Str}_L$ is highly homogeneous if and only if $\mathscr{M} \models \mathfrak{H}\mathfrak{H}$.

Proposition (Combining previously cited results of Ackerman–Freer–Kruckman–Kwiatkowska–P.). Let ϑ be an $\mathscr{L}_{\omega_1\omega}(L)$-sentence.

- If $\vartheta \wedge \neg\mathfrak{H}\mathfrak{H}$ has trivial definable closure, then $I(\vartheta) = 2^{\aleph_0}$.
- If $\vartheta \wedge \neg\mathfrak{H}\mathfrak{H}$ has non-trivial definable closure and $\vartheta \wedge \mathfrak{H}\mathfrak{H}$ is equivalent to the disjunction of Scott sentences for n-many non-isomorphic highly homogeneous structures, where $1 \leq n \leq \aleph_0$, then $I(\vartheta) = n$.

  The converse holds as well.

Extra Credit, Redux. What is the value of $I(\vartheta)$ when ϑ is:

- a disjunction of Scott sentences for $(\mathbb{Z}, \leq)$ and for the pure set $\mathbb{N}$
- a disjunction of Scott sentences for $(\mathbb{Q}, \leq)$ and for the pure set $\mathbb{N}$
- a disjunction of Scott sentences for the Rado graph and for the pure set $\mathbb{N}$

The range of the spectrum function

What values can $I(\vartheta)$ take?

- ϑ
  - has non-trivial dcl
    $I(\vartheta) = 0$
  - has trivial dcl
    $I(\vartheta) > 0$
    - $\vartheta \wedge \neg \mathfrak{H}\mathfrak{H}$ has trivial dcl
      $I(\vartheta) = 2^{\aleph_0}$
    - $\vartheta \wedge \neg \mathfrak{H}\mathfrak{H}$ has non-trivial dcl
      - $\vartheta \wedge \mathfrak{H}\mathfrak{H}$ is equivalent to disjunction of Scott sentences of n-many non-isomorphic h.h. structures, $1 \leq n \leq \aleph_0$
        $I(\vartheta) = n$
      - otherwise
        $\aleph_0 < I(\vartheta) \leq 2^{\aleph_0}$
        Which values in this range can $I(\vartheta)$ take?

Example: $\xi \wedge \neg\mathfrak{H}\mathfrak{H}$ has non-trivial dcl and $I(\xi) = 2^{\aleph_0}$

$L = \{U_i : i < \omega\}$, where each U_i is a unary relation symbol. Define

$$\xi := \bigwedge_{i<\omega} \Big((\forall x)\, U_i(x) \vee (\forall x) \neg U_i(x) \Big)$$

- ⋆ Every countable model of ξ is interdefinable with the pure set $\mathbb{N}$, hence is highly homogeneous. Thus $\xi \wedge \neg\mathfrak{H}\mathfrak{H}$ vacuously has non-trivial definable closure.

- ⋆ There are $2^{\aleph_0}$-many non-isomorphic highly homogeneous models of ξ in Str_L. For each such model $\mathscr{M}$, there is a unique ergodic model of any Scott sentence for $\mathscr{M}$. Hence $I(\xi) = 2^{\aleph_0}$.

What values can $I(\vartheta)$ take?

- ϑ
  - has non-trivial dcl
    $I(\vartheta) = 0$
  - has trivial dcl
    $I(\vartheta) > 0$
    - $\vartheta \wedge \neg \mathfrak{H}\mathfrak{H}$ has trivial dcl
      $I(\vartheta) = 2^{\aleph_0}$
    - $\vartheta \wedge \neg \mathfrak{H}\mathfrak{H}$ has non-trivial dcl
      - $\vartheta \wedge \mathfrak{H}\mathfrak{H}$ is equivalent to disjunction of Scott sentences of n-many non-isomorphic h.h. structures, $1 \leq n \leq \aleph_0$
        $I(\vartheta) = n$
      - otherwise
        $\aleph_0 < I(\vartheta) \leq 2^{\aleph_0}$
        $2^{\aleph_0}$ ✓
        $\aleph_0 < I(\vartheta) < 2^{\aleph_0}$??

Can $\vartheta \wedge \neg\mathfrak{H}\mathfrak{H}$ have non-trivial dcl and $\aleph_0 < I(\vartheta) < 2^{\aleph_0}$?

Answer: No, by Silver's Dichotomy

Proposition. Suppose an $\mathscr{L}_{\omega_1\omega}(L)$-sentence ϑ has fewer than $2^{\aleph_0}$-many highly homogeneous models in Str_L, up to isomorphism. Then ϑ has only countably many highly homogeneous models in Str_L, up to isomorphism.

Proof. Let $\sim$ be the equivalence relation on Str_L given by: $\mathscr{M} \sim \mathscr{N}$ if and only if

- $\mathscr{M}, \mathscr{N} \models \vartheta \wedge \mathfrak{H}\mathfrak{H}$ and $\mathscr{M}, \mathscr{N}$ have the same $\mathscr{L}_{\omega\omega}(L)$ theory; or
- $\mathscr{M}, \mathscr{N} \models \neg(\vartheta \wedge \mathfrak{H}\mathfrak{H})$.

Then $\sim$ is a Borel equivalence relation on Str_L. By Silver's Dichotomy, $\mathrm{Str}_L/\sim$ is either countable or of size $2^{\aleph_0}$. The result follows from hypothesis, as any highly homogeneous structure is $\aleph_0$-categorical.

What values can $I(\vartheta)$ take?

ϑ

- has non-trivial dcl
  $I(\vartheta) = 0$
- has trivial dcl
  $I(\vartheta) > 0$
  - $\vartheta \wedge \neg\mathfrak{H}\mathfrak{H}$ has trivial dcl
    $I(\vartheta) = 2^{\aleph_0}$
  - $\vartheta \wedge \neg\mathfrak{H}\mathfrak{H}$ has non-trivial dcl
    - $\vartheta \wedge \mathfrak{H}\mathfrak{H}$ is equivalent to disjunction of Scott sentences of n-many non-isomorphic h.h. structures, $1 \leq n \leq \aleph_0$
      $I(\vartheta) = n$
    - otherwise
      $I(\vartheta) = 2^{\aleph_0}$

A classification for $I(\vartheta)$

Theorem (Ackerman–Freer–Kruckman–Kwiatkowska–P. 2022+). For an $\mathscr{L}_{\omega_1\omega}(L)$-sentence ϑ, exactly one of the following holds.

$\diamond_0$ ϑ has non-trivial definable closure. In this case, $I(\vartheta) = 0$.

$\diamond_n$ $\vartheta \wedge \neg\mathfrak{H}\mathfrak{H}$ has non-trivial definable closure, and there are sentences ρ_i, $i < n$, where $1 \leq n \leq \aleph_0$, such that the ρ_i are Scott sentences of non-isomorphic highly homogeneous structures in Str_L and

$$\models (\vartheta \wedge \mathfrak{H}\mathfrak{H}) \leftrightarrow \Big(\bigvee_{i<n} \rho_i\Big).$$

In this case, $I(\vartheta) = n$.

$\diamond_{2^{\aleph_0}}$ $I(\vartheta) = 2^{\aleph_0}$.

Analogue of Vaught Conjecture for ergodic structures

Corollary. (Ackerman–Freer–Kruckman–Kwiatkowska–P. 2022+) If an $\mathscr{L}_{\omega_1\omega}(L)$-sentence has fewer than $2^{\aleph_0}$-many ergodic models, then it has countably many ergodic models.

This answers a question asked by C. Freer at the Vaught's Conjecture Workshop held in Berkeley in June, 2015.

Two questions

Q. Range of the spectrum function in a finite language?

The maximal range of the spectrum function is $\{0, 1, \ldots, \aleph_0, 2^{\aleph_0}\}$.

Observe:

- ⋆ The values 0, $2^{\aleph_0}$ can each be achieved in a language with a single relation symbol.
- ⋆ The values n, $1 \leq n \leq \aleph_0$, can each be respectively achieved in a language with n relation symbols.

Q. Can the value $\aleph_0$ be achieved in some finite language?

Observe also:

- ⋆ The maximal range can be achieved in a countably infinite language.

Q. Can the maximal range be achieved in some finite language?

Thank you!