Revisiting the canonical Erdős-Rado theorem

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Outline

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- The finite canonical Erdős-Rado theorem.
- Canonical colorings on Fraïssé structures.
- Results.

Part I

The finite canonical Erdős-Rado theorem

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Theorem (Erdős-Rado, 50)

Let $m \leq n \in \mathbb{N}$, $\chi : \binom{\mathbb{N}}{m} \to \mathbb{N}$.

Then there is $\tilde{B} \in \binom{\mathbb{N}}{n}$ *such that* χ *is canonical on* $\binom{\tilde{B}}{m}$ *i.e.*

$$\exists I \subset m \quad \forall a, a' \in \binom{\tilde{B}}{m} \quad \chi(a) = \chi(a') \Leftrightarrow \mathrm{proj}_I(a) = \mathrm{proj}_I(a')$$

In words: Any coloring is essentially a projection when suitably localized.

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Remark

When $I = \emptyset$, χ *is constant.*

Conversely, $I = \emptyset$ *is the only possible canonization when* χ *has finite range.*

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- *Also proved for finite ordered tournaments and finite posets ordered with linear extensions (Mašulović, 19).*

Question

- *How frequent are such results in structural Ramsey/Fraïssé theory?*
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Goal of today's talk:

- Any finite Ramsey theorem in the Fraïssé context admits a canonical Erdős-Rado counterpart...
- ... But finding out what this counterpart is is not Ramsey theory anymore.
- In addition, it seems that there is not more to it than extreme amenability.
- However, certain canonizations can be expressed at the level of groups.

Part II

Canonical colorings

Definition

Let $m \in \mathbb{N}$. A coloring $\chi : \binom{\mathbb{N}}{m} \to \mathbb{N}$ is canonical when the equivalence relation E_χ it induces on $\binom{\mathbb{N}}{m}$ is S_∞-invariant, where

$$a(gE_\chi)a' \Leftrightarrow (g^{-1}a)E_\chi(g^{-1}a')$$

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Theorem (Erdős-Rado, 50 ; V2)

Let $m \leq n \in \mathbb{N}$. Then:

1. $\forall \chi : \binom{\mathbb{N}}{m} \to \mathbb{N} \quad \exists \tilde{B} \in \binom{\mathbb{N}}{n} \quad \exists c \ \mathit{canonical} \quad \chi \restriction \binom{\tilde{B}}{m} = c \restriction \binom{\tilde{B}}{m}$
2. *Up to a renaming of its range, any canonical coloring is a projection.*

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2. *Up to a renaming of its range, any canonical coloring is a projection.*

It is under that form that the canonical Erdős-Rado theorem will generalize to the Fraïssé context. Possibly, the class of canonical colorings will be larger than just the set of projections.

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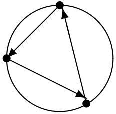
Examples

$\mathbb{N}$, $(\mathbb{Q}, <)$, *the random graph, the generic countable* K_n*-free graph, the countably-dimensional vector space over a given finite field, the countable atomless Boolean algebra, the generic countable poset, the dense local order* $S(2)$*:*

- *Vertices: Rational points of* S^1 *in complex plane (no opposite points).*
- *Arcs:* $x \to y$ *iff (counterclockwise angle from* x *to* y*)* $< \pi$.



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- *$\mathbb{F}$ has the Ramsey property when:*
  for any finite $A, B \subset \mathbb{F}$, any finite coloring of $\binom{\mathbb{F}}{A}$, there is $\tilde{B} \cong B$ where all embeddings of A have same color.

Remark

This really is a property of $\mathrm{Age}(\mathbb{F})$, the set of all finite substructures of $\mathbb{F}$, rather than $\mathbb{F}$.

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- *Now many more by: Aranda et al., Bartosova-Kwiatkowska, Bartosova-Lopez-Abad-Mbombo, Bodirsky, Dorais et al., Foniok, Foniok-Böttcher, Jasiński, Jasiński-Laflamme-NVT-Woodrow, Kechris-Sokić, Kechris-Sokić-Todorcevic, Kwiatkowska, Nešetřil, Nešetřil-Hubička, NVT, Sokić, Solecki, Solecki-Zhao,...*

Part III

Results

Proposition

Let $\mathbb{F}$ be Fraïssé with the Ramsey property, and $A, B \subset \mathbb{F}$ finite. Then:

$$\forall \chi : \binom{\mathbb{F}}{A} \to \mathbb{N} \quad \exists \tilde{B} \cong B \quad \exists c \ \textit{canonical} \quad \chi \upharpoonright \binom{\tilde{B}}{A} = c \upharpoonright \binom{\tilde{B}}{A}$$

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Proof.

Straighforward consequence of the KPT correspondence:

Consider E_χ the equivalence relation induced by χ.

In the cpct space $[2]^{\binom{\mathbb{F}}{A} \times \binom{\mathbb{F}}{A}}$, the subset $\overline{\mathrm{Aut}(\mathbb{F}) \cdot E_\chi}$ is $\mathrm{Aut}(\mathbb{F})$-invariant.

As $\mathbb{F}$ has the Ramsey property, $\mathrm{Aut}(\mathbb{F})$ is extremely amenable, and $\overline{\mathrm{Aut}(\mathbb{F}) \cdot E_\chi}$ contains a fixed point E_c, induced by a coloring c.

Then, up to a renaming of its range, c is as required. □

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- The natural step at that point is... to ask your favorite model theorist... who will tell you that there is no general method for such a task, and that it could be truly difficult.
- Still, there are some natural conditions under which
  - there are only finitely many such relations.
  - the projections are the only canonical colorings.

Definition

Let $A, B \subset \mathbb{F}$ *finite substructures. A* ***joint embedding*** $\langle a, b \rangle$ *of* A *and* B *is an ordered pair of embeddings of* A *and* B *into some finite substructure* $C \subset \mathbb{F}$ *such that* C *is generated by* $a(A) \cup b(B)$*. NB: There is a natural notion of isomorphism between two such objects.*

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Definition

Let $A \subset \mathbb{F}$ be finite. The joint embedding digraph *$\mathcal{G}_A$ is defined as:*

- *Vertex set: $\binom{\mathbb{F}}{A}$.*
- *If $a_0, a_1 \in \binom{\mathbb{F}}{A}$, there is a directed edge from a_0 to a_1, labelled with the isomorphism type of the joint embedding $\langle a_0, a_1\rangle$.*

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Remark

- $\mathrm{Aut}(\mathbb{F})$ *naturally acts on $\mathcal{G}_A$: $a \cdot g = g^{-1} \circ a$*
- *if $(a_0, a_1) \cong (a'_0, a'_1)$ in $\mathcal{G}_A$, $\exists g \in \mathrm{Aut}(\mathbb{F})$ $a_0 \cdot g = a'_0$, $a_1 \cdot g = a'_1$.*
- *The $\mathrm{Aut}(\mathbb{F})$-invariant equivalence relations on $\binom{\mathbb{F}}{A}$ are obtained as unions of various edge relations in $\mathcal{G}_A$.*

Only finitely many canonical colorings

Proposition

Let $\mathbb{F}$ be Fraïssé, $A \subset \mathbb{F}$ finite. Assume that there are only finitely many isomorphism types of joint embeddings of two copies of A. Then:
Up to a renaming of the range, the set of canonical colorings of $\binom{\mathbb{F}}{A}$ is finite.

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Corollary

Assume that $\mathrm{Aut}(\mathbb{F})$ is Roelcke precompact (e.g. $\mathbb{F}$ has finite language, or is $\aleph_0$-categorical).
Then, for every finite $A \subset \mathbb{F}$, and up to a renaming of the range, there are only finitely many canonical colorings of $\binom{\mathbb{F}}{A}$.

Canonical colorings and projections

Theorem

Let $\mathbb{F}$ be Fraïssé ordered structure, satisfying the connecting joint embedding property. Then, up to a renaming of the range, the canonical colorings are exactly the projections.

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The connecting joint embedding property is a combinatorial condition on the joint embedding digraphs $(\mathcal{G}_A)_{A\subset\mathbb{F}}$ which:

- isolates specific joint embedding types.
- ensures that any $\mathrm{Aut}(\mathbb{F})$-invariant equivalence relation on $\binom{\mathbb{F}}{A}$ contains an edge relation of specific type.
- ensures that any such $\mathrm{Aut}(\mathbb{F})$-invariant equivalence relation on $\binom{\mathbb{F}}{A}$ is induced by a projection.

Proposition

Let $\mathbb{F}$ be Fraïssé order expansion of a Fraïssé structure with the free amalgamation property. Then the connecting joint embedding property holds.
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Let $\mathcal{K}$ be Fraïssé, satisfying the free amalgamation property. Then, the class $\mathcal{K} * \mathcal{LO}$ has the Ramsey property.

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Let $\mathcal{K}$ be Fraïssé, satisfying the free amalgamation property. Then, the class $\mathcal{K} * \mathcal{LO}$ has the Ramsey property.

Corollary

Let $\mathcal{K}$ be Fraïssé with free amalgamation, $\mathbb{F} = \mathrm{Flim}(\mathcal{K} * \mathcal{LO})$, $A, B \subset \mathbb{F}$ finite. Then:

$$\forall \chi : \binom{\mathbb{F}}{A} \to \mathbb{N} \quad \exists \tilde{B} \cong B \quad \exists I \subset A \quad \chi \upharpoonright \binom{\tilde{B}}{A} = \mathrm{proj}_I \upharpoonright \binom{\tilde{B}}{A}$$

Examples

- *Total orders (finite canonical Erdős-Rado theorem),*
- *Ordered graphs, ordered K_n-free graphs,*
- *Ordered hypergraphs, ordered hypergraphs forbidding a family of irreducible hypergraphs,*
- *Ordered tournaments (from ordered graphs),*
- *Posets, ordered with linear extensions,*
- *Metric spaces with distances in $S \subset \mathbb{R}_+$ with no jump, i.e. where $(s, 2s] \cap S \neq \emptyset$ whenever $s \in S$ is non-maximal.*

Non-examples

- *Any class with imprimitive action of $\mathrm{Aut}(\mathbb{F})$ on $\mathbb{F}$ e.g. ultrametric spaces.*
- *Finite Boolean algebras.*
- *Finite vector spaces over finite fields.*

Canonical colorings and projections at the level of groups

Theorem

Let $\mathbb{F}$ be a Fraïssé structure. TFAE:

i) *For every finite substructure $A \subset \mathbb{F}$, up to a renaming of the range, the canonical colorings of $\binom{\mathbb{F}}{A}$ are exactly the projections.*

ii) *For every finite substructure $A \subset \mathbb{F}$, every subgroup H of $\mathrm{Aut}(\mathbb{F})$ containing $\mathrm{Stab}(A)$ is of the form $\mathrm{Stab}(A')$ for some substructure $A' \subset A$.*

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Idea of proof: rests on a standard fact in permutation groups:

- The action of $\mathrm{Aut}(\mathbb{F})$ on $\binom{\mathbb{F}}{A}$ defined by $a \cdot g = g^{-1} \circ a$ is transitive, so there is a $1-1$ correspondence between
  - canonical equivalence relations on $\binom{\mathbb{F}}{A}$
  - subgroups H of $\mathrm{Aut}(\mathbb{F})$ such that $\mathrm{Stab}(A) \leq H$.
- Under this correspondence, the equivalence relation induced by $\mathrm{proj}_{A'}$ corresponds to $H = \mathrm{Stab}(A')$.

Corollary

Let $\mathbb{F}$ be a Fraïssé structure where finite substructures have trivial definable closure (e.g. because of strong amalgamation). TFAE:

i) *For every finite substructure $A \subset \mathbb{F}$, up to a renaming of the range, the canonical colorings of $\binom{\mathbb{F}}{A}$ are exactly the projections.*

ii) *Every open subgroup of $\mathrm{Aut}(\mathbb{F})$ is of the form $\mathrm{Stab}(A)$ for some finite substructure A of $\mathbb{F}$.*

A question

Question

When the canonical colorings are the projections, the group $\mathrm{Aut}(\mathbb{F})$ *is topologically simple. What about the converse?*

Remark

When $\mathbb{F}$ *has free amalgamation,* $\mathrm{Aut}(\mathbb{F})$ *is top. simple provided it is not Sym*$(\mathbb{F})$ *and it acts transitively on* $\mathbb{F}$ *(McPherson-Tent, 11).*