Isomorphism of locally compact Polish metric structures

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April 20, 2022
A **structure** is a set $M$ equipped with relations $R_i$, $i \in I$, functions $f_j$, $j \in J$, and constants $c_k$, $k \in K$.

**Examples:**

- graphs $(R, E)$,
- Boolean algebras $(B, \wedge, \vee, −, 0, 1)$,
- metric spaces $(M, \{d_r\}_{r \in R})$, $R \subseteq \mathbb{R}^+$.
Let $L$ be a relational signature $L$, with $n_i$ the arity of relational symbol $R_i$, $i \in I$. Then $\text{Mod}(L) = \prod_{i \in I} 2^{|N|^{n_i}}$ is the space of codes of all countable $L$-structures with universe $\mathbb{N}$. 
The space of countable structures and the logic action

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The group $S_\infty$, acting on $\text{Mod}(L)$ by permuting the universe, induces the isomorphism equivalence relation $\cong$ on $\text{Mod}(L)$. In particular, Vaught transforms can be used:
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For open $U \subseteq S_\infty$, and $A \subseteq \text{Mod}(L)$

$$M \in A^*_U \iff \forall^* g \in U \ g. M \in A.$$
$\mathcal{L}_{\omega_1 \omega}$ and its fragments

We will work in the setting of infinitary logic $\mathcal{L}_{\omega_1 \omega}$, i.e., an extension of the finitary logic $\mathcal{L}_{\omega \omega}$ allowing for countably infinite conjunctions $\bigwedge_i \phi_i$, and disjunctions $\bigvee_i \phi_i$. 
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A (countable) fragment $F$ is a countable set of $L_{\omega_1 \omega}$-formulas containing all $L_{\omega \omega}$-formulas, and closed under $\land$, $\lor$, $\neg$, and $\exists$. We can talk about $F$-theories, $F$-types, type spaces $S_n(T)$, spaces $\text{Mod}(T) \subseteq \text{Mod}(L)$ of models of a theory $T$, isomorphism relations $\cong_T$ on $\text{Mod}(T)$, etc.
\( \mathcal{L}_{\omega_1\omega} \) and its fragments

We will work in the setting of infinitary logic \( \mathcal{L}_{\omega_1\omega} \), i.e., an extension of the finitary logic \( \mathcal{L}_{\omega\omega} \) allowing for countably infinite conjunctions \( \bigwedge_i \phi_i \), and disjunctions \( \bigvee_i \phi_i \).

A (countable) **fragment** \( F \) is a countable set of \( \mathcal{L}_{\omega_1\omega} \)-formulas containing all \( \mathcal{L}_{\omega\omega} \)-formulas, and closed under \( \land, \lor, \neg \), and \( \exists \). We can talk about \( F \)-theories, \( F \)-types, type spaces \( S_n(T) \), spaces \( \text{Mod}(T) \subseteq \text{Mod}(L) \) of models of a theory \( T \), isomorphism relations \( \cong_T \) on \( \text{Mod}(T) \), etc.

The space \( S_n(T) \) of all \( n \)-\( F \)-types is equipped with the logic topology \( \tau_n \) with basis consisting of sets \([ \phi ]\), defined by \( \text{tp}(\bar{a}) \in [ \phi ] \) iff \( \phi^M(\bar{a}) = 1 \), where \( \phi \in F \), \( M \in \text{Mod}(T) \), \( \bar{a} \) is a tuple in \( M \).

In a similar fashion, we can define a topology \( t_F \) on \( \text{Mod}(L) \).
An equivalence relation $E$ on a Polish space $X$ is (Borel) reducible to an equivalence relation $F$ on a Polish space $Y$ if there is a Borel mapping $f : X \to Y$ such that, for any $x_1, x_2 \in X$,

$$x_1 \ E \ x_2 \iff f(x_1) \ F \ f(x_2).$$
Complexity of equivalence relations

An equivalence relation \( E \) on a Polish space \( X \) is \textbf{(Borel)} reducible to an equivalence relation \( F \) on a Polish space \( Y \) if there is a Borel mapping \( f : X \to Y \) such that, for any \( x_1, x_2 \in X \),

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\]

Important types of equivalence relations:

- smooth, i.e., reducible to the identity,
- essentially countable, i.e., reducible to a relation with countable classes,
- classifiable by countable structures, i.e., reducible to the isomorphism relation on a Borel class of countable structures.
$\mathcal{L}_{\omega_1\omega}$ and descriptive set theory

**Theorem (Lopez-Escobar)**

Let $L$ be a signature. Every isomorphism-invariant Borel set $A \subseteq \text{Mod}(L)$ is of the form $\text{Mod}(T)$ for some countable theory $T \subseteq \mathcal{L}_{\omega_1\omega}$.
\( \mathcal{L}_{\omega_1 \omega} \) and descriptive set theory

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**Theorem (Hjorth-Kechris)**

Let \( T \) be a countable theory, and let \( \cong_T \) be the isomorphism relation on \( \text{Mod}(T) \). TFAE:

1. \( \cong_T \) is essentially countable,
2. there exists a fragment \( F \) such that for every \( M \in \text{Mod}(T) \), there is a tuple \( \bar{a} \) such that \( \text{Th}_F(M, \bar{a}) \) is \( \aleph_0 \)-categorical.
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**Theorem (Hjorth-Kechris)**

Let $T$ be a countable theory, and let $\equiv_T$ be the isomorphism relation on $\text{Mod}(T)$. TFAE:

1. $\equiv_T$ is essentially countable,

2. there exists a fragment $F$ such that for every $M \in \text{Mod}(T)$, there is a tuple $\bar{a}$ such that $\text{Th}_F(M, \bar{a})$ is $\aleph_0$-categorical.

**Corollary**

Isomorphism of finitely generated countable groups is essentially countable.
A **metric structure** is a complete and bounded metric space $(M, d)$ equipped with bounded uniformly continuous functions $R_i : M^{n_i} \to \mathbb{R}$, $i \in I$ (relations), uniformly continuous functions $f_j : M^{n_j} \to M$, $j \in J$, and constants $c_k$, $k \in K$.

A metric signature consists of relation (including the metric), function, and constant symbols, as well as arities, moduli of continuity $\Delta : [0, +\infty)^n \to [0, +\infty)$, and bounds $I \subseteq \mathbb{R}$ for relation symbols. Each of the relations and functions of a metric structure in a given signature must respect its modulus of continuity. Each of the relations must respect its bound.
Metric structures

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**Examples:**

- Complete metric spaces \((M, d)\);
- Measure algebras \((B, d, \wedge, \vee, 0, 1)\);
- Banach spaces, \(C^*\)-algebras, etc.
Let $L$ be a countable relational signature $L$, with $n_i$ the arity of relation $R_i$, $i \in I$, where $R_0 = d$. Then $\text{Mod}(L) \subseteq \prod_{i \in I} \mathbb{R}^{\mathbb{N}^{n_i}}$ is the space of codes of all Polish metric structures with universe containing $\mathbb{N}$ as a (tail-)dense subset of $M$. 

Remark: No Vaught transforms. However, for $M \in \text{Mod}(L)$, let $D \subseteq M^\mathbb{N}$ be the Polish space of all tail-dense sequences in $M$, and $\pi: D \to [M]$ a natural projection from $D$ onto the isomorphism class $[M]$ of $M$. For $A \subseteq \text{Mod}(L)$, $\bar{a} \in \mathbb{N}^\prec \mathbb{N}$, and $u \in \mathbb{Q}^+$, put $M \in A^* \bar{a}, u \iff \forall^* y \in B_D(M) < u(\bar{a})(\pi(y) \in A)$. 

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$$M \in A^{*\bar{a},u} \leftrightarrow \forall^* y \in B_{<u}^{D(M)}(\bar{a})(\pi(y) \in A),$$
Continuous $\mathcal{L}_{\omega\omega}$ and $\mathcal{L}_{\omega_1\omega}$

Formulas of continuous finitary logic $\mathcal{L}_{\omega\omega}$ are defined using

- continuous functions $s : [a, b]^n \to [a, b]$ as connectives. Alternatively: polynomials or just $\{0, 1, \frac{x}{2}, \cdot, +, -\}$,

- inf and sup as quantifiers.
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  Alternatively: polynomials or just $\{0, 1, \frac{x}{2}, \cdot, +, -\}$,
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Analogs of infinite conjunctions and disjunctions in the continuous infinitary logic $\mathcal{L}_{\omega_1\omega}$ are defined with $\inf_i \phi_i, \sup_i \phi_i$ as infinitary connectives, provided that all $\phi_i$ respect a single modulus of continuity and bound.
Type spaces

For a given fragment $F$, and $F$-theory $T$, the type $p = \text{tp}(\bar{a})$ of $\bar{a}$ in $M \in \text{Mod}(T)$ is the family of all conditions of the form $\phi(\bar{x}) = r$ such that $\phi^M(\bar{a}) = r$. We write $p(\phi) = r$. 

The (Polish) logic topology $\tau$ on $S_n(T)$ is defined by sets $[\phi < r]$ of all the types $p$ such that $p(\phi) = s$ for some $s < r$. We can also define $t_F$ on $\text{Mod}(L)$. There is also a natural (complete) metric $\partial$ on $S_n(T)$. For $F = L_{\omega\omega}$, it can be defined by $\partial(p, q) = \inf \{d_M(\bar{a}, \bar{b}) : M \models T, \bar{a}, \bar{b} \in M^n, \text{tp}(\bar{a}) = p, \text{tp}(\bar{b}) = q\}$. In general, $\partial(p, q) = \sup_{\phi \in F_1} |p(\phi) - q(\phi)|$, where $F_1$ are 1-Lipschitz formulas.
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$$\partial(p, q) = \inf \{ d^M(\bar{a}, \bar{b}) : M \models T, \, \bar{a}, \bar{b} \in M^n, \, \text{tp}(\bar{a}) = p, \text{tp}(\bar{b}) = q \}$$
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In general,

$$\partial(p, q) = \sup_{\phi \in F_1} |p(\phi) - q(\phi)|,$$

where $F_1$ are 1-Lipschitz formulas.
Continuous $\mathcal{L}_{\omega_1\omega}$ and descriptive set theory

Theorem (Ben Yaacov-Doucha-Nies-Tsankov)

*Every isomorphism-invariant Borel set $A \subseteq \text{Mod}(L)$ is of the form $\text{Mod}(T)$ for some (countable) theory $T \subseteq \mathcal{L}_{\omega_1\omega}$.*
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Theorem (Hallbäck, M., Tsankov)

Let $T$ be a theory with locally compact Polish models. TFAE:

1. $\cong_T$ is essentially countable,
2. there exists a fragment $F$ such that for every $M \in \text{Mod}(T)$, there is $k \in \mathbb{N}$ such that the set

   \[ \{ \bar{a} \in M^k : \text{Th}_F(M, \bar{a}) \text{ is } \aleph_0\text{-rigid} \} \]

   has non-empty interior in $M^k$.  

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Corollary (Kechris)

Every orbit equivalence relation induced by a locally compact Polish group is essentially countable.
Theorem (M.)

Let $T$ be a countable theory with locally compact models. Then $\cong_T$ is classifiable by countable structures.
\( \alpha \)-AE families

Let \( \beta = 0 \) or a limit ordinal.
$\alpha$-AE families

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▸ An $(-1)$-AE family $P(\bar{x})$ is a formula $\phi(\bar{x})$ in $F$. 

Moreover, every $\alpha$-AE family $P(\bar{x}) = \{ p_{k,l}(\bar{x}_{k,l}) \}$, $\alpha \geq 1$, comes equipped with a fixed $u_{P} \geq 0$ such that $u_{P} \geq u_{p_{k,l}}$, $k, l \in \mathbb{N}$. 

Let $\beta = 0$ or a limit ordinal.

- An $(-1)$-AE family $P(\bar{x})$ is a formula $\phi(\bar{x})$ in $F$.
- A $\beta$-AE family $P(\bar{x})$ is a collection of $\gamma$-AE families $p_k(\bar{x})$, $k \in \mathbb{N}, \gamma < \beta$. 
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- A \((\beta + 1)\)-AE family \(P(\bar{x})\) is a collection of \(\gamma\)-AE families \(p_{k,l}(\bar{x}_{k,l})\), \(\gamma < \beta\), \(k, l \in \mathbb{N}\), \(\bar{x} \subseteq \bar{x}_{k,l}\).
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- A $(\beta + n)$-AE family $P(\bar{x})$, $2 \leq n < \omega$, is a collection of $(\beta + n - 2)$-AE families $p_{k,l}(\bar{x}_{k,l})$, $k, l \in \mathbb{N}$, $\bar{x} \subseteq \bar{x}_{k,l}$. 
\textbf{\(\alpha\)-AE families}

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  \item A \(\beta\)-AE family \(P(\bar{x})\) is a collection of \(\gamma\)-AE families \(p_k(\bar{x})\), \(k \in \mathbb{N}, \gamma < \beta\).
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  \item A \((\beta + n)\)-AE family \(P(\bar{x})\), \(2 \leq n < \omega\), is a collection of \((\beta + n - 2)\)-AE families \(p_{k,l}(\bar{x}_{k,l})\), \(k, l \in \mathbb{N}, \bar{x} \subseteq \bar{x}_{k,l}\).
\end{itemize}

Moreover, every \(\alpha\)-AE family \(P(\bar{x}) = \{p_{k,l}(\bar{x}_{k,l})\}\), \(\alpha \geq 1\), comes equipped with a fixed \(u_P \geq 0\) such that \(u_P \geq u_{p_{k,l}}, k, l \in \mathbb{N}\).
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- \((\beta + n)\)-AE family \(P(\bar{x}) = \{p_{k,l}(\bar{x}_{k,l})\}, 1 \leq n < \omega\), if it holds in \(M\) that

\[
\forall \bar{b} \in B_{\uparrow P}^M(\bar{a}) \forall \bar{v} > 0 \forall k \exists \bar{c} \in B_{\downarrow \bar{v}}^M(\bar{b}) \exists l (\bar{c} \text{ realizes } p_{k,l}(\bar{x}_{k,l}) \text{ in } M).
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\[\forall \bar{b} \in B_u^M(\bar{a}) \forall v > 0 \forall k \exists \bar{c} \in B_v^M(\bar{b}) \exists l (\bar{c} \text{ realizes } p_{k,l}(\bar{x}_{k,l}) \text{ in } M).\]

Remark: For a countable \(M\),

\[\forall \bar{b} \supseteq \bar{a} \forall k \exists \bar{c} \supseteq \bar{b} \exists l (\bar{c} \text{ realizes } p_{k,l}(\bar{x}_{k,l}) \text{ in } M).\]
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- $(-1)$-AE family $P(\bar{x}) = \phi(\bar{a})$ if $\phi^M(\bar{a}) = 0$,
- $\beta$-AE family $P(\bar{x})$ if it realizes every $p(\bar{x}) \in P(\bar{x})$,
- $(\beta + n)$-AE family $P(\bar{x}) = \{p_{k,l}(\bar{x}_{k,l})\}$, $1 \leq n < \omega$, if it holds in $M$ that

$$\forall \bar{b} \in B^M_{u_p} (\bar{a}) \forall v > 0 \forall k \exists \bar{c} \in B^M_{v} (\bar{b}) \exists l (\bar{c} \text{ realizes } p_{k,l}(\bar{x}_{k,l}) \text{ in } M).$$

**Remark:** For a countable $M$,

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If $\emptyset$ in $M$ realizes $P(\emptyset)$, we say that $M$ **models** $P$. 
AE families and Borel complexity

Let $F$ be fragment in signature $L$, and let $2 \leq \alpha < \omega_1$. Let $m = 1$ if $\alpha < \omega$, and $m = 0$ otherwise.

**Theorem**

Suppose that $A \in \Pi^0_\alpha(t_F)$ for some $A \subseteq \text{Mod}(L)$. For every $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $u \in \mathbb{Q}^+$, there exists an $(\alpha - m)$-AE family $P(\bar{x})$ such that

$$A^{*\bar{a},u} = \{ N \in \text{Mod}(L) : \bar{a} \text{ realizes } P(\bar{x}) \text{ in } N \}.$$

**Corollary**

Suppose that $[M] \in \Pi^0\alpha(t_F)$ for some $M \in \text{Mod}(L)$. There exists an $(\alpha - m)$-AE family $P_M$ such that

$$[M] = \{ N \in \text{Mod}(L) : N \text{ models } P_M \}.$$
Locally compact structures

For a theory $T$, locally compact $M \in \text{Mod}(T)$, $n \in \mathbb{N}$, and $n$-tuple $\bar{a}$ in $M$, let

$$\rho(\bar{a}) = \sup\{r \in \mathbb{R} : \overline{B_{\leq r}^M(\bar{a})} \text{ is compact}\},$$

$$\Theta_n(M) = \{\text{tp}(\bar{b}) : \bar{b} \in M^n\}.$$
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Fix a countable basis $\mathcal{U}_n = \{U_{l,n} \}$ for each $\tau_n$, and put $\mathcal{U} = \bigcup_n \mathcal{U}_n$. For $U \in \mathcal{U}_n$, and $\epsilon > 0$, $(U, \epsilon)$ is $\bar{a}$-good in $M$ if

- $\text{tp}(\bar{a}) \in U$,
- $2\epsilon < \rho(\bar{a})$,
- there is $\delta > 0$ such that $U \cap B_{<2\epsilon}(\text{tp}(\bar{a})) \subseteq B_{<\epsilon-\delta}(\text{tp}(\bar{a}))$. 

For every $\delta > 0$ there exist $U \in \mathcal{U}$ and $0 < \epsilon < \delta$ such that $(U, \epsilon)$ is $\bar{a}$-good,

if $(U, \epsilon)$ is $\bar{a}$-good, then

$$\overline{B_{<\epsilon}(tp(\bar{a})) \cap U^T} \subseteq \Theta_{|\bar{a}|}(M),$$

if $(U, \epsilon)$ is $\bar{a}$-good, there is $\delta > 0$ such that $d(\bar{a}, \bar{a}') < \delta$ implies that $(U, \epsilon)$ is $\bar{a}'$-good, and

$$U \cap B_{<\epsilon}(tp(\bar{a})) = U \cap B_{<\epsilon}(tp(\bar{a}')).$$
Locally compact structures

For \( \bar{a} \in \mathbb{N}^{<\mathbb{N}}, U \in \mathcal{U}_n, \) and \( \epsilon \in \mathbb{Q}^+ \), define

\[
T^0_{U,\epsilon}(\bar{a}) = B_{<\epsilon}(\text{tp}(\bar{a})) \cap U^\tau,
\]

if \( (U, \epsilon) \) is \( \bar{a} \)-good,

\[
T^0_{U,\epsilon}(\bar{a}) = \emptyset,
\]

otherwise, and

\[
T^\alpha_{U,\epsilon}(\bar{a}) = \{ T^\beta_{U',\epsilon'}(\bar{a}') : \beta < \alpha, |\bar{a}'| \geq |\bar{a}|, U' \in \mathcal{U}_{|\bar{a}'|}, U' \upharpoonright |\bar{a}| \subseteq U, \epsilon' \leq \epsilon \}
\]

for \( \alpha > 0 \). Also, for \( u > 0 \), put

\[
T^\alpha_u(\bar{a}) = \{ T^\beta_u(v(\bar{b})) : \beta < \alpha, \bar{b} \in B_{\bar{u}}^{<\omega}(\bar{a}), |\bar{b}| \geq |\bar{a}|, U \in \mathcal{U}_{|\bar{b}|}, v > 0 \},
\]

\[
T^\alpha(M) = T^\alpha_1(\emptyset).
\]
Remark: For a countable $M$, and $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, put

$$tp^0(\bar{a}) = tp(\bar{a}),$$

$$tp^\alpha(\bar{a}) = \{ tp^\beta(\bar{b}) : \beta < \alpha, \bar{b} \in \mathbb{N}^{<\mathbb{N}}, \bar{a} \subseteq \bar{b} \},$$

$$Th^\alpha(M) = tp^\alpha(\emptyset).$$
Theorem

Let $F$ be a fragment, and let $T$ be an $F$-theory. Suppose that $M, N \in \text{Mod}(T)$ are locally compact, and $T^\alpha_u(\bar{a}) = T^\alpha_u(\bar{a}')$ for some tuples $\bar{a}, \bar{a}'$ in $M, N$, respectively. Then every $\alpha$-AE family $P(\bar{x})$ with $u_P \leq u$ realized by $\bar{a}'$, is also realized by $\bar{a}$.
Locally compact structures

Theorem
Let $F$ be a fragment, and let $T$ be an $F$-theory. Suppose that $M, N \in \text{Mod}(T)$ are locally compact, and $T^\alpha_u(\bar{a}) = T^\alpha_u(\bar{a}')$ for some tuples $\bar{a}, \bar{a}'$ in $M, N$, respectively. Then every $\alpha$-AE family $P(\bar{x})$ with $u_P \leq u$ realized by $\bar{a}'$, is also realized by $\bar{a}$.

Theorem
Let $F$ be a fragment, and let $T$ be an $F$-theory with locally compact models. Suppose that $[M] \in \Pi^0_\alpha(t_F)$, $\alpha \geq 2$, for some $M \in \text{Mod}(T)$. Let $m = 1$ if $\alpha < \omega$, and $m = 0$ otherwise. Then

$$[M] = \{N \in \text{Mod}(T) : T^{\alpha-m}(N) = T^{\alpha-m}(M)\}.$$
Locally compact structures

Theorem

Let $T$ be a countable theory with locally compact models. Then $\cong_T$ is classifiable by countable structures.

For $M \in \operatorname{Mod}(T)$, $C_M$ consists of elements

$$x = (B_\epsilon(tp(\bar{a})) \cap U^T, |\bar{a}|, U, \epsilon),$$

where $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, $U \in \mathcal{U}_{|\bar{a}|}$, $\epsilon \in \mathbb{Q}^+$, and $(U, \epsilon)$ is $\bar{a}$-good, and relations $O_l$, $R_{k,l,\delta}$, $k, l \in \mathbb{N}$, $\delta \in \mathbb{Q}^+$, and $E$, defined as follows:

- $O_l(x)$ iff $U_{l,|\bar{a}|} \cap B_\epsilon(tp(\bar{a})) \cap U^T = \emptyset$,
- $R_{k,l,\delta}(x)$ iff $k = |\bar{a}|$, $U = U_{l,n}$, $\delta = \epsilon$,
- $xEx'$ iff $|\bar{a}'| \geq |\bar{a}|$, $U' \upharpoonright |\bar{a}| \subseteq U$, $\epsilon' \leq \epsilon$. 
Theorem (M.)

Isometry of locally compact Polish metric spaces is Borel reducible to graph isomorphism.
Isometry of locally compact Polish metric spaces

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Isometry of locally compact Polish metric spaces is Borel reducible to graph isomorphism.

A locally compact Polish metric space \((K, d)\), regarded as an element \(\mathcal{K}(\mathcal{U})\) of the hyperspace of Urysohn space, can be coded in a Borel way as \(M_K \in \text{Mod}(L)\) with the trivial signature \(L\), and metric bounded by 1: using the Kuratowski–Ryll-Nardzewski theorem, pick a countable tail-dense subset of \(K\), and replace \(d\) with \(1/(1 + d)\).
Borel isomorphism relations

A relation $E$ on a standard Borel space $X$ is potentially $\Pi^0_\alpha$ if there is a Polish topology $t$ inducing the Borel structure of $X$, and such that $E \in \Pi^0_\alpha(t \times t)$.

For $\alpha < \omega_1$, $P^0(\mathbb{N}) = \mathbb{N}$, $P^\alpha(\mathbb{N}) =$ all countable subsets of $P^{<\alpha}(\mathbb{N}) \cup \mathbb{N}$, where $P^{<\alpha}(\mathbb{N}) = \bigcup_{\beta < \alpha} P^\beta(\mathbb{N})$, and $=^\alpha$ is the equality on $P^\alpha(\mathbb{N})$.

**Theorem (Hjort, Kechris, Louveau)**

Let $F$ be a fragment in the classical $L_{\omega_1\omega}$, and let $T$ be an $F$-theory. If $\equiv_T$ is potentially $\Pi^0_{\alpha+2}$, where $\alpha \geq 1$, then $\equiv_T$ is Borel reducible to $=^\alpha+1$. 
A relation $E$ on a standard Borel space $X$ is potentially $\Pi^0_\alpha$ if there is a Polish topology $t$ inducing the Borel structure of $X$, and such that $E \in \Pi^0_\alpha(t \times t)$.

For $\alpha < \omega_1$, $\mathcal{P}^0(\mathbb{N}) = \mathbb{N}$, $\mathcal{P}^\alpha(\mathbb{N}) =$ all countable subsets of $\mathcal{P}^{<\alpha}(\mathbb{N}) \cup \mathbb{N}$, where $\mathcal{P}^{<\alpha}(\mathbb{N}) = \bigcup_{\beta < \alpha} \mathcal{P}^\beta(\mathbb{N})$, and $=_{\alpha}$ is the equality on $\mathcal{P}^\alpha(\mathbb{N})$.

**Theorem (Hjort, Kechris, Louveau)**

Let $F$ be a fragment in the classical $\mathcal{L}_{\omega_1\omega}$, and let $T$ be an $F$-theory. If $\cong_T$ is potentially $\Pi^0_{\alpha+2}$, where $\alpha \geq 1$, then $\cong_T$ is Borel reducible to $=_{\alpha+1}$.

**Theorem (M.)**

Let $F$ be a fragment in the continuous $\mathcal{L}_{\omega_1\omega}$, and let $T$ be an $F$-theory with locally compact models. If $\cong_T$ is potentially $\Pi^0_{\alpha+2}$, where $\alpha \geq 1$, then $\cong_T$ is Borel reducible to $=_{\alpha+1}$. 
Borel isomorphism relations

**Theorem**

Let $L$ be a signature, let $t$ be a Polish topology on $\text{Mod}(L)$ consisting of Borel subsets of the standard topology, and let $\alpha < \omega_1$. There exists a fragment $F$ such that $A^{*\bar{a},1/k} \in \Pi_0^0(t_F)$ for every $A \in \Pi_0^0(t)$, $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $k > 0$.

**Corollary**

Let $L$ be a signature, and let $T$ be a theory such that $\equiv_T$ is potentially $\Pi_0^0$. There exists a fragment $F$ such that $[M] \in \Pi_0^0(t_F)$ for every $M \in \text{Mod}(T)$. 
Borel isomorphism relations

For a fragments $F$, $F'$, and a formula $\phi$,

- $\text{rk}_F(\phi) = 0$ if $\phi \in F$,
- $\text{rk}_F(\phi) = \sup\{\text{rk}_F(\phi_i) + 1\}$ if $\phi = \lor_i \phi_i$ or $\phi = \land_i \phi_i$,
- $\text{rk}_F(\phi) = \text{rk}_F(\psi)$ if $\phi$ is in the fragment gen. by $F$ and $\psi$,
- $\text{rk}_F(F') = \sup\{\text{rk}_F(\phi) : \phi \in F'\}$.

Remark: $\phi$ can be coded as an element of $\mathcal{P}^{\omega}(\mathbb{N})$ if $\text{rk}_F(\phi) \leq \alpha$. 
Borel isomorphism relations

For a fragments $F$, $F'$, and a formula $\phi$,

$\triangleright$ \( \text{rk}_F(\phi) = 0 \) if $\phi \in F$,

$\triangleright$ \( \text{rk}_F(\phi) = \sup\{\text{rk}_F(\phi_i) + 1\} \) if $\phi = \bigvee_i \phi_i$ or $\phi = \bigwedge_i \phi_i$,

$\triangleright$ \( \text{rk}_F(\phi) = \text{rk}_F(\psi) \) if $\phi$ is in the fragment gen. by $F$ and $\psi$,

$\triangleright$ \( \text{rk}_F(F') = \sup\{\text{rk}_F(\phi) : \phi \in F'\} \).

Remark: $\phi$ can be coded as an element of $\mathcal{P}^\alpha(\mathbb{N})$ if $\text{rk}_F(\phi) \leq \alpha$.

Theorem

Let $F$ be a fragment, and let $T$ be an $F$-theory with locally compact models. Suppose that $[M] \in \Pi_{\alpha+2}(t_F)$ for some $M \in \text{Mod}(T)$, $\alpha \geq 1$. There is a fragment $F_M \supseteq F$ such that $[M] \in \Pi^0_2(t_{F_M})$, and $\text{rk}_F(F_M) = \alpha$. 
Theorem (Hallbäck, M., Tsankov)

Let $F$ be a fragment and let $T$ be an $F$-theory. For any $M \in \text{Mod}(T)$, $[M]$ is $G_\delta$ in the topology $t_F$ iff $M$ is an atomic model of $\text{Th}_F(M)$.

Lemma (Tsankov)

Let $L$ be a signature. For every fragment $F$, there exists a fragment $F' \supseteq F$ such that if $M \in \text{Mod}(L)$ is $F$-atomic, then $\text{Th}_{F'}(M)$ is $\aleph_0$-categorical.
Thank You!