Isomorphism of locally compact Polish metric structures

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Structures

A **structure** is a set M equipped with relations R_i, $i \in I$, functions f_j, $j \in J$, and constants c_k, $k \in K$.

Examples:

- graphs (R, E),
- Boolean algebras $(B, \wedge, \vee, -, 0, 1)$,
- metric spaces $(M, \{d_r\}_{r \in R})$, $R \subseteq \mathbb{R}^+$.

The space of countable structures and the logic action

Let L be a relational signature L, with n_i the arity of relational symbol R_i, $i \in I$. Then $\mathrm{Mod}(L) = \prod_{i \in I} 2^{\mathbb{N}^{n_i}}$ is the space of codes of all countable L-structures with universe $\mathbb{N}$.

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The group S_∞, acting on $\mathrm{Mod}(L)$ by permuting the universe, induces the isomorphism equivalence relation $\cong$ on $\mathrm{Mod}(L)$. In particular, Vaught transforms can be used:

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For open $U \subseteq S_\infty$, and $A \subseteq \mathrm{Mod}(L)$

$$M \in A^{*U} \Leftrightarrow \forall^* g \in U\, g.M \in A.$$

$\mathcal{L}_{\omega_1\omega}$ and its fragments

We will work in the setting of infinitary logic $\mathcal{L}_{\omega_1\omega}$, i.e., an extension of the finitary logic $\mathcal{L}_{\omega\omega}$ allowing for countably infinite conjunctions $\bigwedge_i \phi_i$, and disjunctions $\bigvee_i \phi_i$.

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A (countable) **fragment** F is a countable set of $\mathcal{L}_{\omega_1\omega}$-formulas containing all $\mathcal{L}_{\omega\omega}$-formulas, and closed under $\wedge$, $\vee$, $\neg$, and $\exists$. We can talk about F-theories, F-types, type spaces $S_n(T)$, spaces $\mathrm{Mod}(T) \subseteq \mathrm{Mod}(L)$ of models of a theory T, isomorphism relations $\cong_T$ on $\mathrm{Mod}(T)$, etc.

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The space $S_n(T)$ of all n-F-types is equipped with the logic topology τ_n with basis consisting of sets $[\phi]$, defined by $\mathrm{tp}(\bar{a}) \in [\phi]$ iff $\phi^M(\bar{a}) = 1$, where $\phi \in F$, $M \in \mathrm{Mod}(T)$, $\bar{a}$ is a tuple in M.

In a similar fashion, we can define a topology t_F on $\mathrm{Mod}(L)$.

Complexity of equivalence relations

An equivalence relation E on a Polish space X is (**Borel**) **reducible** to an equivalence relation F on a Polish space Y if there is a Borel mapping $f : X \to Y$ such that, for any $x_1, x_2 \in X$,

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Important types of equivalence relations:

- smooth, i.e., reducible to the identity,
- essentially countable, i.e., reducible to a relation with countable classes,
- classifiable by countable structures, i.e., reducible to the isomorphism relation on a Borel class of countable structures.

$\mathcal{L}_{\omega_1\omega}$ and descriptive set theory

Theorem (Lopez-Escobar)

Let L be a signature. Every isomorphism-invariant Borel set $A \subseteq \mathrm{Mod}(L)$ is of the form $\mathrm{Mod}(T)$ for some countable theory $T \subseteq \mathcal{L}_{\omega_1\omega}$.

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Theorem (Hjorth-Kechris)

Let T be a countable theory, and let $\cong_T$ be the isomorphism relation on $\mathrm{Mod}(T)$. TFAE:

1. *$\cong_T$ is essentially countable,*
2. *there exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, there is a tuple $\bar{a}$ such that $\mathrm{Th}_F(M, \bar{a})$ is $\aleph_0$-categorical.*

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Corollary

Isomorphim of finitely generated countable groups is essentially countable.

Metric structures

A **metric structure** is a complete and bounded metric space (M, d) equipped with bounded uniformly continuous functions $R_i : M^{n_i} \to \mathbb{R}$, $i \in I$ (relations), uniformly continuous functions $f_j : M^{n_j} \to M$, $j \in J$, and constants c_k, $k \in K$.

A metric signature consists of relation (including the metric), function, and constant symbols, as well as arities, moduli of continuity $\Delta : [0, +\infty)^n \to [0, +\infty)$, and bounds $I \subseteq \mathbb{R}$ for relation symbols. Each of the relations and functions of a metric structure in a given signature must respect its modulus of continuity. Each of the relations must respect its bound.

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Examples:

- Complete metric spaces (M, d);
- Measure algebras $(B, d, \wedge, \vee, 0, 1)$;
- Banach spaces, C^*-algebras, etc.

The space of Polish metric structures

Let L be a countable relational signature L, with n_i the arity of relation R_i, $i \in I$, where $R_0 = d$. Then $\mathrm{Mod}(L) \subseteq \prod_{i \in I} \mathbb{R}^{\mathbb{N}^{n_i}}$ is the space of codes of all Polish metric structures with universe containing $\mathbb{N}$ as a (tail-)dense subset of M.

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Remark: No Vaught transforms. However, for $M \in \mathrm{Mod}(L)$, let $D \subseteq M^{\mathbb{N}}$ be the Polish space of all tail-dense sequences in M, and $\pi : D \to [M]$ a natural projection from D onto the isomorphism class $[M]$ of M. For $A \subseteq \mathrm{Mod}(L)$, $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $u \in \mathbb{Q}^+$, put

$$M \in A^{*\bar{a},u} \Leftrightarrow \forall^* y \in B^{D(M)}_{<u}(\bar{a})(\pi(y) \in A),$$

Continuous $\mathcal{L}_{\omega\omega}$ and $\mathcal{L}_{\omega_1\omega}$

Formulas of continuous finitary logic $\mathcal{L}_{\omega\omega}$ are defined using

- continuous functions $s : [a, b]^n \to [a, b]$ as connectives. Alternatively: polynomials or just $\{0, 1, \frac{x}{2}, \cdot, +, -\}$,
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Analogs of infinite conjunctions and disjunctions in the continuous infinitary logic $\mathcal{L}_{\omega_1\omega}$ are defined with $\inf_i \phi_i$, $\sup_i \phi_i$ as infinitary connectives, provided that all ϕ_i respect a single modulus of continuity and bound.

Type spaces

For a given fragment F, and F-theory T, the type $p = \mathrm{tp}(\bar{a})$ of $\bar{a}$ in $M \in \mathrm{Mod}(T)$ is the family of all conditions of the form $\phi(\bar{x}) = r$ such that $\phi^M(\bar{a}) = r$. We write $p(\phi) = r$.

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The (Polish) logic topology τ on $S_n(T)$ is defined by sets $[\phi < r]$ of all the types p such that $p(\phi) = s$ for some $s < r$. We can also define t_F on $\mathrm{Mod}(L)$.

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There is also a natural (complete) metric ∂ on $S_n(T)$. For $F = \mathcal{L}_{\omega\omega}$, it can be defined by

$$\partial(p, q) = \inf\{d^M(\bar{a}, \bar{b}) : M \models T,\ \bar{a}, \bar{b} \in M^n,\ \mathrm{tp}(\bar{a}) = p, \mathrm{tp}(\bar{b}) = q\ \}$$

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In general,

$$\partial(p, q) = \sup_{\phi \in F_1} |p(\phi) - q(\phi)|,$$

where F_1 are 1-Lipschitz formulas.

Continuous $\mathcal{L}_{\omega_1\omega}$ and descriptive set theory

Theorem (Ben Yaacov-Doucha-Nies-Tsankov)

Every isomorphism-invariant Borel set $A \subseteq \mathrm{Mod}(L)$ *is of the form* $\mathrm{Mod}(T)$ *for some (countable) theory* $T \subseteq \mathcal{L}_{\omega_1\omega}$.

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Theorem (Hallbäck, M., Tsankov)

Let T be a theory with locally compact Polish models. TFAE:

1. $\cong_T$ *is essentially countable,*
2. *there exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, there is $k \in \mathbb{N}$ such that the set*

$$\{\bar{a} \in M^k : \mathrm{Th}_F(M, \bar{a}) \text{ is } \aleph_0\text{-rigid}\}$$

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Corollary (Kechris)

Every orbit equivalence relation induced by a locally compact Polish group is essentially countable.

Isomorphism of locally compact Polish metric structures

Theorem (M.)

Let T be a countable theory with locally compact models. Then $\cong_T$ is classifiable by countable structures.

α-AE families

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Moreover, every α-AE family $P(\bar{x}) = \{p_{k,l}(\bar{x}_{k,l})\}$, $\alpha \geq 1$, comes equipped with a fixed $u_P \geq 0$ such that $u_P \geq u_{p_{k,l}}$, $k, l \in \mathbb{N}$.

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If $\emptyset$ in M realizes $P(\emptyset)$, we say that M **models** P.

AE families and Borel complexity

Let F be fragment in signature L, and let $2 \leq \alpha < \omega_1$. Let $m = 1$ if $\alpha < \omega$, and $m = 0$ otherwise.

Theorem

Suppose that $A \in \mathbf{\Pi}^0_\alpha(t_F)$ for some $A \subseteq \mathrm{Mod}(L)$. For every $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $u \in \mathbb{Q}^+$, there exists an $(\alpha - m)$-AE family $P(\bar{x})$ such that

$$A^{*\bar{a},u} = \{N \in \mathrm{Mod}(L) : \bar{a} \textit{ realizes } P(\bar{x}) \textit{ in } N\}.$$

Corollary

Suppose that $[M] \in \mathbf{\Pi}^0_\alpha(t_F)$ for some $M \in \mathrm{Mod}(L)$. There exists an $(\alpha - m)$-AE family P_M such that

$$[M] = \{N \in \mathrm{Mod}(L) : N \textit{ models } P_M\}.$$

Locally compact structures

For a theory T, locally compact $M \in \mathrm{Mod}(T)$, $n \in \mathbb{N}$, and n-tuple $\bar{a}$ in M, let

$$\rho(\bar{a}) = \sup\{r \in \mathbb{R} : \overline{B^{M^n}_{<r}(\bar{a})} \text{ is compact}\},$$

$$\Theta_n(M) = \{\mathrm{tp}(\bar{b}) : \bar{b} \in M^n\}.$$

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Fix a countable basis $\mathcal{U}_n = \{U_{l,n}\}$ for each τ_n, and put $\mathcal{U} = \bigcup_n \mathcal{U}_n$. For $U \in \mathcal{U}_n$, and $\epsilon > 0$, (U, ϵ) is *$\bar{a}$-good* in M if

- $\mathrm{tp}(\bar{a}) \in U$,
- $2\epsilon < \rho(\bar{a})$,
- there is $\delta > 0$ such that $U \cap B_{<2\epsilon}(\mathrm{tp}(\bar{a})) \subseteq B_{<\epsilon-\delta}(\mathrm{tp}(\bar{a}))$.

Locally compact structures

- For every $\delta > 0$ there exist $U \in \mathcal{U}$ and $0 < \epsilon < \delta$ such that (U, ϵ) is $\bar{a}$-good,
- if (U, ϵ) is $\bar{a}$-good, then

$$\overline{B_{<\epsilon}(\mathrm{tp}(\bar{a})) \cap U}^{\tau} \subseteq \Theta_{|\bar{a}|}(M),$$

- if (U, ϵ) is $\bar{a}$-good, there is $\delta > 0$ such that $d(\bar{a}, \bar{a}') < \delta$ implies that (U, ϵ) is $\bar{a}'$-good, and

$$U \cap B_{<\epsilon}(\mathrm{tp}(\bar{a})) = U \cap B_{<\epsilon}(\mathrm{tp}(\bar{a}')).$$

Locally compact structures

For $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, $U \in \mathcal{U}_n$, and $\epsilon \in \mathbb{Q}^+$, define

$$T^0_{U,\epsilon}(\bar{a}) = \overline{B_{<\epsilon}(\mathrm{tp}(\bar{a})) \cap U}^{\tau},$$

if (U, ϵ) is $\bar{a}$-good,

$$T^0_{U,\epsilon}(\bar{a}) = \emptyset,$$

otherwise, and

$$T^\alpha_{U,\epsilon}(\bar{a}) = \{\, T^\beta_{U',\epsilon'}(\bar{a}') : \beta < \alpha, |\bar{a}'| \geq |\bar{a}|, U' \in \mathcal{U}_{|\bar{a}'|}, U' \upharpoonright |\bar{a}| \subseteq U, \epsilon' \leq \epsilon\}$$

for $\alpha > 0$. Also, for $u > 0$, put

$$T^\alpha_u(\bar{a}) = \{\, T^\beta_{U,v}(\bar{b}) : \beta < \alpha, \bar{b} \in B^{M^{<\omega}}_u(\bar{a}), |\bar{b}| \geq |\bar{a}|, U \in \mathcal{U}_{|\bar{b}|}, v > 0\},$$

$$T^\alpha(M) = T^\alpha_1(\emptyset).$$

Locally compact structures

Remark: For a countable M, and $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, put

$$\mathrm{tp}^0(\bar{a}) = \mathrm{tp}(\bar{a}),$$

$$\mathrm{tp}^\alpha(\bar{a}) = \{\mathrm{tp}^\beta(\bar{b}) : \beta < \alpha,\ \bar{b} \in \mathbb{N}^{<\mathbb{N}},\ \bar{a} \subseteq \bar{b}\},$$

$$\mathrm{Th}^\alpha(M) = \mathrm{tp}^\alpha(\emptyset).$$

Locally compact structures

Theorem

Let F be a fragment, and let T be an F-theory. Suppose that $M, N \in Mod(T)$ are locally compact, and $T^{\alpha}_{u}(\bar{a}) = T^{\alpha}_{u'}(\bar{a}')$ for some tuples $\bar{a}$, $\bar{a}'$ in M, N, respectively. Then every α-AE family $P(\bar{x})$ with $u_P \leq u$ realized by $\bar{a}'$, is also realized by $\bar{a}$.

Locally compact structures

Theorem

Let F be a fragment, and let T be an F-theory. Suppose that $M, N \in Mod(T)$ are locally compact, and $T^{\alpha}_{u}(\bar{a}) = T^{\alpha}_{u'}(\bar{a}')$ for some tuples $\bar{a}$, $\bar{a}'$ in M, N, respectively. Then every α-AE family $P(\bar{x})$ with $u_P \leq u$ realized by $\bar{a}'$, is also realized by $\bar{a}$.

Theorem

Let F be a fragment, and let T be an F-theory with locally compact models. Suppose that $[M] \in \mathbf{\Pi}^0_{\alpha}(t_F)$, $\alpha \geq 2$, for some $M \in \mathrm{Mod}(T)$. Let $m = 1$ if $\alpha < \omega$, and $m = 0$ otherwise. Then

$$[M] = \{N \in \mathrm{Mod}(T) : T^{\alpha - m}(N) = T^{\alpha - m}(M)\}.$$

Locally compact structures

Theorem

Let T be a countable theory with locally compact models. Then $\cong_T$ is classifiable by countable structures.

For $M \in \mathrm{Mod}(T)$, C_M consists of elements

$$x = (\overline{B_\epsilon(\mathrm{tp}(\bar{a})) \cap U}^\tau, |\bar{a}|, U, \epsilon),$$

where $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, $U \in \mathcal{U}_{|\bar{a}|}$, $\epsilon \in \mathbb{Q}^+$, and (U, ϵ) is $\bar{a}$-good, and relations O_l, $R_{k,l,\delta}$, $k, l \in \mathbb{N}$, $\delta \in \mathbb{Q}^+$, and E, defined as follows:

- $O_l(x)$ iff $U_{l,|\bar{a}|} \cap \overline{B_\epsilon(\mathrm{tp}(\bar{a})) \cap U}^\tau = \emptyset$,
- $R_{k,l,\delta}(x)$ iff $k = |\bar{a}|$, $U = U_{l,n}$, $\delta = \epsilon$,
- xEx' iff $|\bar{a}'| \geq |\bar{a}|$, $U' \restriction |\bar{a}| \subseteq U$, $\epsilon' \leq \epsilon$.

Isometry of locally compact Polish metric spaces

Theorem (M.)

Isometry of locally compact Polish metric spaces is Borel reducible to graph isomorphism.

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A locally compact Polish metric space (K, d), regarded as an element $\mathcal{K}(\mathbb{U})$ of the hyperspace of Urysohn space, can be coded in a Borel way as $M_K \in \mathrm{Mod}(L)$ with the trivial signature L, and metric bounded by 1: using the Kuratowski–Ryll-Nardzewski theorem, pick a countable tail-dense subset of K, and replace d with $1/(1+d)$.

Borel isomorphism relations

A relation E on a standard Borel space X is **potentially $\mathbf{\Pi}^0_\alpha$** if there is a Polish topology t inducing the Borel structure of X, and such that $E \in \mathbf{\Pi}^0_\alpha(t \times t)$.

For $\alpha < \omega_1$, $\mathcal{P}^0(\mathbb{N}) = \mathbb{N}$, $\mathcal{P}^\alpha(\mathbb{N}) =$ all countable subsets of $\mathcal{P}^{<\alpha}(\mathbb{N}) \cup \mathbb{N}$, where $\mathcal{P}^{<\alpha}(\mathbb{N}) = \bigcup_{\beta<\alpha} \mathcal{P}^\beta(\mathbb{N})$, and $=_\alpha$ is the equality on $\mathcal{P}^\alpha(\mathbb{N})$.

Theorem (Hjort, Kechris, Louveau)

Let F be a fragment in the classical $\mathcal{L}_{\omega_1\omega}$, and let T be an F-theory. If $\cong_T$ is potentially $\mathbf{\Pi}^0_{\alpha+2}$, where $\alpha \geq 1$, then $\cong_T$ is Borel reducible to $=_{\alpha+1}$.

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Theorem (M.)

Let F be a fragment in the continuous $\mathcal{L}_{\omega_1\omega}$, and let T be an F-theory with locally compact models. If $\cong_T$ is potentially $\mathbf{\Pi}^0_{\alpha+2}$, where $\alpha \geq 1$, then $\cong_T$ is Borel reducible to $=_{\alpha+1}$.

Borel isomorphism relations

Theorem

Let L be a signature, let t be a Polish topology on $\mathrm{Mod}(L)$ consisting of Borel subsets of the standard topology, and let $\alpha < \omega_1$. There exists a fragment F such that $A^{*\bar{a},1/k} \in \mathbf{\Pi}^0_\alpha(t_F)$ for every $A \in \mathbf{\Pi}^0_\alpha(t)$, $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $k > 0$.

Corollary

Let L be a signature, and let T be a theory such that $\cong_T$ is potentially $\mathbf{\Pi}^0_\alpha$. There exists a fragment F such that $[M] \in \mathbf{\Pi}^0_\alpha(t_F)$ for every $M \in \mathrm{Mod}(T)$.

Borel isomorphism relations

For a fragments F, F', and a formula ϕ,

- $\mathrm{rk}_F(\phi) = 0$ if $\phi \in F$,
- $\mathrm{rk}_F(\phi) = \sup\{\mathrm{rk}_F(\phi_i) + 1\}$ if $\phi = \bigvee_i \phi_i$ or $\phi = \bigwedge_i \phi_i$,
- $\mathrm{rk}_F(\phi) = \mathrm{rk}_F(\psi)$ if ϕ is in the fragment gen. by F and ψ,
- $\mathrm{rk}_F(F') = \sup\{\mathrm{rk}_F(\phi) : \phi \in F'\}$.

Remark: ϕ can be coded as an element of $\mathcal{P}^\alpha(\mathbb{N})$ if $\mathrm{rk}_F(\phi) \leq \alpha$.

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Remark: ϕ can be coded as an element of $\mathcal{P}^\alpha(\mathbb{N})$ if $\mathrm{rk}_F(\phi) \leq \alpha$.

Theorem

Let F be a fragment, and let T be an F-theory with locally compact models. Suppose that $[M] \in \mathbf{\Pi}^0_{\alpha+2}(t_F)$ for some $M \in \mathrm{Mod}(T)$, $\alpha \geq 1$. There is a fragment $F_M \supseteq F$ such that $[M] \in \mathbf{\Pi}^0_2(t_{F_M})$, and $\mathrm{rk}_F(F_M) = \alpha$.

Borel isomorphism relations

Theorem (Hallbäck, M., Tsankov)

Let F be a fragment and let T be an F-theory. For any $M \in Mod(T)$, $[M]$ is G_δ in the topology t_F iff M is an atomic model of $\mathrm{Th}_F(M)$.

Lemma (Tsankov)

Let L be a signature. For every fragment F, there exists a fragment $F' \supseteq F$ such that if $M \in \mathrm{Mod}(L)$ is F-atomic, then $\mathrm{Th}_{F'}(M)$ is $\aleph_0$-categorical.

Thank You!